M.Sc. PHYSICS – I YEAR

DKP12 : MATHEMATICAL PHYSICS

SYLLABUS

UNIT I : VECTOR SPACE AND MATRICES

Linear independence of vectors – Dimension – Basis – Inner product of two vectors – Properties of inner product – Schmidt's orthonormalization method – Linear transformations – Matrices – inverse of a matrix – orthogonal matrix – unitary matrix – eigen value and eigen vectors of a matrix – Diagonalisation – Cayley Hamilton Theorem.

UNIT II : FUNCTIONS AND POLYNOMIALS

Beta , Gamma functions – Dirac delta function and its properties – Green's function – Bessel differential equation – Generating function for $J_n(x)$ – Recurrence relation for $J_n(x)$ – Legendre differential equation – Generating function for $P_n(x)$ – Recurrence relation for $P_n(x)$- Hermite differential equation – Generating function for $H_n(x)$ – Recurrence relation for $H_n(x)$

UNIT III : FOURIER AND LAPLACE TRANSFORM

Fourier transform-properties of Fourier transform-convolution – Fourier cosine and sine transform-Fourier transform of derivatives- Application of Fourier transform-vibrations in a string-Laplace transform-inverse Laplace transform- Application of Laplace transform-Simple Harmonic motion

UNIT IV : COMPLEX ANALYSIS

Complex variables- complex conjugate and modulus of a complex number-algebraic operations of complex numbers-function of a complex variable-analytic function-Cauchy-Riemann equation in polar form-line integral of a complex function-Cauchy integral theorem-Cauchy integral formula-Derivatives of an analytic function

UNIT V : GROUP THEORY

Concept of a group-Group multiplication table of order 2, 3, 4 groups- Group symmetry of equilateral triangle- Group symmetry of a square-permutation group-conjugate elements-representation through similarity transformation-reducible and irreducible representation-SU(2) group-SO(2) group.

UNIT I : VECTOR SPACE AND MATRICES

Linear independence of vectors – Dimension – Basis – Inner product of two vectors – Properties of inner product – Schmidt's orthonormalization method – Linear transformations – Matrices – inverse of a matrix – orthogonal matrix – unitary matrix – eigen value and eigen vectors of a matrix – Diagonalisation – Cayley Hamilton Theorem.

1.1 Vector Space:

Let $(F, +, \cdot)$ be a field of real numbers and let V be a set together with an operation of addition (+) and a scalar multiplication (·). The operations of (+) and (·) on elements of V by elements of F yielding again elements of V and satisfying the algebraic laws, then the set V will be called a real"$Vector\ space\ or\ Vector\ space\ over\ F$"

ie.,$V_n(R)$ is a vector space over R.

If the components of n-dimensional vectors are rational numbers then $V_n(Q)$ is a vector space over rational number.

$n - tuples(C_1, C_2, \ldots \ldots C_n)$of complex numbers together with addition and scalar multiplication would form a vector space over complex numbers.

1.2 Linear Independence of Vectors:

Let V be a vector space over a field F and let $S = \{\alpha_1, \alpha_2, \ldots \ldots \ldots . \alpha_k)$ be a finite sub set of V. Then S is said to be linearly independent if and only if every equations of the form

$$a_1\alpha_1 + a_2\alpha_2 + \cdots \ldots \ldots . + a_k\alpha_k = 0, \qquad a_i \in F \implies a_1 = a_2 = \cdots a_k = 0$$

ie., an infinite set T is said to be linearly independent if and only if every finite sub set of T is linearly independent.

(eg) Any set which consists of a single non-zero vector is independent.

Any set of vectors which is not linearly independent is linearly dependent. Any vector which contains 0 is dependent. Two vectors A and B are said to depend on each other when one of them can be expressed in term of the second.

ie., $A = k\,B$ where k is a non-zero scalar or $c\,A + d\,B = 0$

Where $c \ \ and \ \ d$ are non-zero scalar constants.

The dependence is said to be linear when the vectors in the expression are of degree one.

Exercise:

1. Show that the vectors (1,2,3), (2,2,0) forms linearly independent.

Solution: Given $\alpha_1 = (1,2,3)$; $\alpha_2 = (2,2,0)$

$a_1(1,2,3) + a_2(2,2,0) = 0$

$(a_1, 2a_1, 3a_1) + (2a_2, 2a_2, 0) = 0$

$a_1 + 2a_2 = 0$

$2a_1 + 2a_2 = 0$

$3a_1 = 0$; $a_1 = 0$

Then $0 + 2a_2 = 0$; $a2 = 0$

Both a_1, and a_2 are zero therefore the given vectors are linearly independent.

2. Show that the four vectors $\alpha_1 = (1,1,0,1)$, $\alpha_2 = (1,0,0,2)$, $\alpha_3 = (0,1,2,-3)$, and $\alpha_4 = (1,1,1,1)$ are linearly independent.

Solution:

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$

$c_1(1,1,0,1) + c_2(1,0,0,2) + c_3(0,1,2,-3) + c_4(1,1,1,1) = 0$

$c_1 + c_2 + c_4 = 0$; $c_1 + c_3 + c_4 = 0$; $2c_3 + c_4 = 0$; $c_1 + 2c_2 - 3c_3 + c_4 = 0$

On solving these equations we get $c_1 = c_2 = c_3 = c_4 = 0$. Therefore the vectors α_1, α_2, α_3, α_4 are linearly independent.

1.3 Basis:

In three dimensional space, $(a_1, a_2, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$(a_1, a_2, a_3) = a_1 \epsilon_1 + a_2 \epsilon_2 + a_3 \epsilon_3$ where ϵ_i, $i = 1,2,3$ represents the triple whose i^{th} component is one and whose other components are zero. $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ spans the space and is linearly independent. These 3 vectors are unit vectors along 3 coordinate axes. This set is a maximal linearly independent sub set of the space. A second maximal linearly independent set in 3 dimensional space is $\beta_1 = (0,1,1)$; $\beta_2 = (1,0,1)$; $\beta_3 = (1,1,0)$.

A maximal linearly independent sub set of a vector space V is called a basis of V. Since a basis for V, spans V, every vector ξ is a linear combination of vectors of V.

$$\xi = C_1\alpha_1 + C_2\alpha_2 + \dots\dots + C_k\alpha_k$$

The Scalars in this representation are Unique.

ie., If $\xi = b_1\alpha_1 + b_2\alpha_2 + \dots\dots + b_k\alpha_k$,

then $\xi - \xi = (C_1 - b_1)\alpha_1 + (C_2 - b_2)\alpha_2 + \dots\dots + (C_k - b_k)\alpha_k$

$$0 = (C_1 - b_1)\alpha_1 + (C_2 - b_2)\alpha_2 + \ldots\ldots\ldots\ldots + (C_k - b_k)\alpha_k$$

ie., $(C_1 - b_1) = 0$; $(C_2 - b_2) = 0$; …… $(C_k - b_k) = 0$,

And ∴ $C_1 = b_1$; $C_2 = b_2$;⋯⋯ $C_k = b_k$.

This proves that the scalars in the representation are unique.

If a given set of vectors $\phi_1, \phi_2, \phi_3, \cdots\cdots \phi_n$ has the following two properties,

1. The vectors $\phi_1, \phi_2, \phi_3, \cdots\cdots \phi_n$ are linearly independent and
2. Every vector ϕ in the space can be expressed as a linear combination of $\phi_1, \phi_2, \ldots \phi_n$,

then the set $(\phi_1, \phi_2, \phi_3, \cdots\cdots \phi_n)$ is a basis for the vector space.

1.4 Dimensions:

A vector space is said to be n dimensional if it has a finite basis consist of n elements. A vector space with no finite basis is said to be infinite dimensional. The maximum number of linearly independent vectors cannot be more than that of the number of dimensions of the space. That is, dimensionality of a space is the maximum number of linearly independent vectors in the space.

Example:

In a three dimensional space α_1, α_2, α_3 are three linearly independent vectors, and then any other vector ψ in the space can be expressed as

$$\psi = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$

Where c_1, c_2, c_3 are constants, and at least one of them is non-zero.

1.5 Inner Product (Scalar Product):

In ordinary three dimensional space the scalar product achieves,

1. The scalar product of a vector with itself helps to define the length of the vector.
2. It is a measure of relative orientation of the vectors, when the lengths are known.

In a linear vector space the inner product of two vectors ψ and ϕ is denoted by (ψ, ϕ). The inner product has the following properties.

1.6 Properties of Inner Product:

1. $(\psi, \phi + \xi) = (\psi, \phi) + (\psi, \xi)$
2. $(\psi + \phi, \xi) = (\psi, \xi) + (\phi, \xi)$
3. $(\psi, \psi) > 0$ unless $\psi = 0$
4. $(\psi, \phi) = (\phi, \psi)^*$

5. $(\psi, \alpha \phi + \beta \xi) = \alpha (\psi, \phi) + \beta(\psi, \xi)$ Where α and β are arbitrary complex numbers.
6. The norm (length) is denoted by $|| \psi||$, and is defined as $|| \psi|| = (\psi, \psi)^{1\backslash 2}$

In an n- dimensional space, elements of basis are α_1, α_2, ……. $\alpha_{n,}$ (the magnitude of each element of the basis is unity then the elements are called unit vectors) then two vectors ψ and ϕ in the space can be expressed as

$\Psi = c_1\alpha_1 + c_2 \alpha_2 + \ldots\ldots\ldots\ldots. + c_n \alpha_n$ and $\phi = b_1\alpha_1 + b_2 \alpha_2 + \ldots\ldots\ldots\ldots. + b_n \alpha_n$

Then the inner product of ψ and ϕ is

$$(\Psi, \phi) = \sum_{i=1}^{n} c_i^* b_i = c_1^* b_1 + c_2^* b_2 + \ldots\ldots\ldots\ldots. + c_n^* b_n$$

<u>Example</u>

1. Calculate inner product of the two vectors A and B given by

$A = 5 \alpha_1 - 3 \alpha_2 - 4 \alpha_3 - \alpha_4 + 2 \alpha_5$ and $B = - \alpha_1 + 2 \alpha_2 - 3 \alpha_3 + \alpha_4 + \alpha_5$

<u>Solution:</u>

The inner product of A and B is

$(A, B) = (5)(-1) + (-3)(2) + (-4)(-3) + (-1)(1) + (2)(1)$

$= -5 - 6 + 12 - 1$

$= 2$

2. Find the norm of a vector 3i + 4j + 5k

<u>Solution:</u>

Let the vector $\psi = 3i + 4j + 5k$,

Then $(\psi, \psi) = (3i + 4j +5k)\cdot(3i + 4j +5k)$

$= 9 + 16 + 25$

$= 50$

$|| \psi|| = (\psi, \psi)^{1\backslash 2} = (50)^{1\backslash 2}$

<u>1.7 Orthogonal Vector</u>:

If the inner product of two vectors equal to zero, ie., $(\psi_i, \psi_k) = 0$ for $\begin{cases} i \pm k; i, k = 1, 2 \ldots n \\ \psi_i \pm 0, \psi_k \pm 0 \end{cases}$

Then the vectors are said to form an orthogonal set. If the norm within the orthogonal set is unity then the set is called orthonormal set, ie., $|| \psi_i|| = 1$ & $|| \psi_k|| = 1$

1.8 Orthonormal basis:

The norm of vector ϕ in a linear vector space is defined to be a positive real number associated with ϕ such that the following properties are satisfied

1. $|| \phi || \geq 0$
2. $|| \phi || = 0$ only if $\phi = 0$
3. $|| \alpha \phi || = | \alpha | \, || \phi ||$ where α is an arbitrary complex numbers.
4. $|| \phi_1 + \phi_2 || \leq || \phi_1 || + || \phi_2 ||$

To Prove,

$|| \phi_1 + \phi_2 || \leq || \phi_1 || + || \phi_2 ||$

The positive square root of scalar product of a vector with itself can be taken as the norm of the vector.

ie., $|| \phi || = (\phi, \phi)^{1/2}$ and $|| \phi ||^2 = (\phi, \phi)$

$$
\begin{aligned}
|| \phi_1 + \phi_2 ||^2 &= (\phi_1 + \phi_2 , \phi_1 + \phi_2) \\
&= (\phi_1 + \phi_2 , \phi_1) +(\phi_1 + \phi_2 , \phi_2) \\
&= (\phi_1 , \phi_1) + (\phi_2 , \phi_1) + (\phi_1 , \phi_2) + (\phi_2 , \phi_2) \\
&= || \phi_1||^2 + (\phi_1 , \phi_2)^* + (\phi_1 , \phi_2) + || \phi_2||^2 \\
&= || \phi_1||^2 +|| \phi_2||^2 + 2\mathrm{Re}(\phi_1 , \phi_2)
\end{aligned}
$$

Since $|(\phi_1 , \phi_2)| \geq \mathrm{Re}(\phi_1 , \phi_2)$

$|| \phi_1 + \phi_2 ||^2 \leq || \phi_1||^2 +|| \phi_2||^2 + 2|(\phi_1 , \phi_2)|$

Using Schwartz inequality, $|| \phi_1 + \phi_2 ||^2 \leq || \phi_1||^2 +|| \phi_2||^2 + 2|| \phi_1|| \, || \phi_2||$

$$\leq (|| \phi_1|| +|| \phi_2||)^2$$

Taking Square root on both sides we get $|| \phi_1 + \phi_2 || \leq || \phi_1 || + || \phi_2 ||$, Hence proved.

The vector of unit norm is said to be normalized. If a vector ϕ is not normalized a normalized vector can be obtained by dividing ϕ by $(\phi ,\phi)^{1/2}$. This process is known as normalization.

Problem:

1. Show that the following two vectors are orthogonal to each other

$A = 4 \alpha_1 - 2 \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_6$ and $B = 2 \alpha_1 + 2 \alpha_3 - 3\alpha_4 + 2 \alpha_5$ where α's form a orthonormal set.

Solution:

The inner product of A and B is

(A, B) =(4)(2) + (-2)(0) + (-1)(2) + (2)(-3) + (0)(2) + (1)(0) = 0

Then (A, A) = (4)(4) + (-2)(-2) + (-1)(-1) + (2)(2) + (1)(1) =26 ;

The norm of A is $(A, A)^{1/2} = (26)^{1/2}$

And The norm of B is (B, B) = (2)(2) + (2)(2) + (-3)(-3) + (2)(2) =21 ;

The norm of B is $(B, B)^{1/2} = (21)^{1/2}$

Therefore the vectors A and B are orthogonal.

1.9 Gram Schmidt orthogonaliationProces:

Let $\psi_1, \psi_2, \psi_3, \ldots\ldots, \psi_n$ be a set of n linearly independent vectors. By Gram Schmidt orthogonalization process we can construct n mutually orthogonal vectors $\phi_1', \phi_2', \phi_3', \ldots\ldots, \phi_n'$ from the given linearly independent vectors. If the constructed vectors are then normalized we get an orthonormal set $\phi_1, \phi_2, \phi_3, \ldots\ldots, \phi_n$.

Construct a set of orthogonal vectors $\phi_1', \phi_2', \phi_3', \ldots\ldots, \phi_n'$ by choosing the following manner

$\phi_1' = \psi_1$

$\phi_2' = \psi_2 + C_{21}\,\phi_1'$

$\phi_3' = \psi_3 + \phi_2' + C_{31}\,\phi_1'$

.

.

.

$\phi_n' = \psi_n + C_{n,n-1}\,\phi_{n-1}' + C_{n,n-2}\,\phi_{n-2}' + \ldots\ldots + \phi_1'$

The coefficient C_{21} , C_{32} , C_{32} , C_{31} , ……….$C_{n,n-1}$, $C_{n,1}$ are chosen such that $\phi_1', \phi_2', \phi_3', \ldots\ldots, \phi_n'$ are mutually orthogonal to each other.

$(\phi_1', \phi_2') = 0$

$(\phi_1', \psi_2 + C_{21}\,\phi_1') = 0$

$(\phi_1', \psi_2) + C_{21}(\phi_1', \phi_1') = 0$

$C_{21}(\phi_1', \phi_1') = -(\phi_1', \psi_2)$

$$C_{21} = -(\phi_1', \psi_2) / (\phi_1', \phi_1')$$

And $(\phi_1', \phi_3') = 0$

$(\phi_1', \psi_3 + C_{32}\,\phi_2' + C_{31}\,\phi_1') = 0$

$(\phi_1', \psi_3) + C_{32}(\phi_1', \phi_2') + C_{31}(\phi_1', \phi_1') = 0$

$(\phi_1', \psi_3) + C_{31}(\phi_1', \phi_1') = 0$

since ϕ_1', ϕ_2' are mutually orthogonal to each other $(\phi_1', \phi_2') = 0$

$C_{31}(\phi_1', \phi_1') = -(\phi_1', \psi_3)$

$$C_{31} = -(\phi_1', \psi_3) / (\phi_1', \phi_1')$$

Similarly $(\phi_2', \phi_3') = 0$

$(\phi_2', \psi_3 + C_{32}\, \phi_2' + C_{31}\, \phi_1') = 0$

$(\phi_2', \psi_3) + C_{32}(\phi_2', \phi_2') + C_{31}(\phi_2', \phi_1') = 0$

$(\phi_2', \psi_3) + C_{32}(\phi_2', \phi_2') = 0$ since $(\phi_2', \phi_1') = 0$

$$C_{32} = -(\phi_2', \psi_3) / (\phi_2', \phi_2')$$

In general $C_{ij} = -(\phi_j', \psi_i) / (\phi_j', \phi_j')$

Using the coefficients we get the orthogonal set ϕ_1', ϕ_2', ϕ_3', …………, ϕ_n' and then we divide each of the vectors by its magnitude then a set of orthonormal vectors is obtained.

Problem:

Using Gram Schmidt orthogonal process, construct an orthogonal set from the linearly independent set of n-tuple

$\Psi_1 = (1,0,0,……0)$; $\Psi_2 = (1,1,0,……0)$; $\Psi_3 = (1,1,1,……0)$;…………………$\Psi_n = (1,1,1,……1)$

Solution:

Let $\phi_1' = \Psi_1 = (1,0,0,……0)$

$\phi_2' = \Psi_2 + C_{21}\, \phi_1'$

$C_{21} = -(\phi_1', \psi_2) / (\phi_1', \phi_1')$

$= -(1,0,0,….0) / (1,0,0,….0)$

$= -1$

$\therefore\ \phi_2' = (1,1,0,……0) - 1\,(1,0,0,……0)$

$\phi_2' = (1,1,0,……0) - (1,0,0,……0)$

$\underline{\phi_2' = (0,1,0,……0)}$

then $\phi_3' = \Psi_3 + C_{32}\, \phi_2' + C_{31}\, \phi_1'$

$C_{31} = -(\phi_1', \psi_3) / (\phi_1', \phi_1')$

$= -(1,0,0,……0) / (1,0,0,……0)$

$= -1$

$C_{32} = -(\phi_2', \psi_3) / (\phi_2', \phi_2')$

$= -(0,1,0,……0) / (0,1,0,……0)$

$= -1$

$\phi_3' = (1,1,1,0,\ldots\ldots\ldots0) - 1\,(0,1,0,\ldots\ldots0) - 1\,(1,0,0,\ldots\ldots0)$

$\phi_3' = (0,0,1,0,\ldots\ldots0)$

similarly we can get

$\phi_4' = (0,0,0,1,\ldots\ldots0)$,

………………………

$\phi_n' = (0,0,0,0,\ldots\ldots1)$

1.10 Linear Transformation:

Let a set of orthonormal vectors $\alpha_1, \alpha_2 \ldots\ldots\alpha_n$ constitutes a basis in an n-dimensional space. Then a vector in the space is expressed as

$$\phi = x_1\,\alpha_1 + x_2\,\alpha_2 + \ldots\ldots + x_n\,\alpha_n$$

where $x_1, x_2, \ldots\ldots x_n$ are the components of vector ϕ along the axes of the coordinate system.

And consider another set of orthonormal vectors $\beta_1, \beta_2, \ldots\ldots, \beta_n$ constitutes another basis in the space. The same vector is expressed as

$$\phi = y_1\,\beta_1 + y_2\,\beta_2 + \ldots\ldots + y_n\,\beta_n$$

where $y_1, y_2, \ldots\ldots y_n$ are the components of the vector ϕ along the axes of the another coordinate system.

To transform the component of the given vector in one coordinate system into another system, the following conditions will be considered. If the origin of the two coordinate systems is same, the transformation is homogeneous; whereas the origins are different the transformation is inhomogeneous.

When components of a vector in one coordinate system can be expressed as a linear combination of the other system, then the transformation is linear. For linear transformation, the components of second coordinate system y^s can be expressed in terms of the component of the first one x^s as

$$Y_i = C_{i1}\,x_1 + C_{i2}\,x_2 + \ldots\ldots + C_{in}x_n = \sum_{j=1}^{n} c_{ij}\,x_j \qquad 1 \le i \le n$$

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & . & c_{1n} \\ c_{21} & c_{22} & . & . & c_{2n} \\ . & & & & \\ . & & & & \\ c_{n1} & c_{n2} & & & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

$$ie., \boldsymbol{Y} = \boldsymbol{C}\,\boldsymbol{X}$$

The transformation matrix C depends on the two coordinate systems and does not depend on components of a vector in the two systems.

$$Inversely \qquad \boldsymbol{X} = \boldsymbol{DY} \quad where\ \boldsymbol{D} = \boldsymbol{C^{-1}}$$

1.11 Matrices:

A matrix may be defined as a square or rectangular array of numbers or functions that obey certain laws. The individual numbers or functions of the array are called the elements of the matrix. And a matrix consists of certain rows (horizontal array) and certain columns (vertical array).

Example $\begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}$, $\begin{pmatrix} a & b & c \\ e & f & g \\ x & y & z \end{pmatrix}$, $\begin{pmatrix} 1-i & 2+i \\ 4+2i & x+iy \\ 2+4i & 3-2i \end{pmatrix}$

and the array function $\begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_4(x) & f_5(x) & f_6(x) \end{pmatrix}$

In the second example (a,b,c) is the first row, (e,f,g) and (x,y,z) are the second and third rows respectively. Similarly (a,e,x), (b,f,y),(c,g,z) are first, second, third columns respectively.

A matrix consists of 'm' rows and 'n' columns is said to be the matrix of order m × n.

$$\text{ie.,}\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{m1} & a_{m2} & & & a_{mn} \end{pmatrix}$$

$A = (a_{ij})_{m\times n}$ means A is a matrix of order (m×n) whose ij^{th} element is a_{ij}, the letter i designates the row and j designates the column to which the element a_{ij} belongs.

1.12 Properties of matrices:

1.12.1 Equality:

Two matrices A and B are equal if and only if they have the same order (m×n) and each element of A is equal to the corresponding element of B. ie.,$a_{ij} = b_{ij}$for all i and j

1.12.2 Addition and Subtraction:

Two matrices of same order of (m×n) can only be added or subtracted.

A + B = C means $a_{ij} + b_{ij} = c_{ij}$ for all i and j

A - B = d means $a_{ij} - b_{ij} = d_{ij}$ for all i and j

Commutative law A + B = B + A

Associative law A +(B + C)= (A + B) + C

Distributive law λ(A + B) = λ A + λ B

1.12.3 Multiplication:

Two matrices A and B can be multiplied in the order of AB only when the order of column of matrix A is same as the order of row of matrix B.

ie., $(A)_{m\times h} \times (B)_{h\times n} = (C)_{m\times n}$

Its elements are given by $C_{ij} = \sum_{k=1}^{h} a_{ik}b_{kj}$

For h = 3,$C_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}\, b_{3j}$

Example :

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \text{and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \text{Find AB}$$

$$(A)_{3\times2} \times (B)_{2\times2} = (C)_{3\times2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ a_{31}b_{11}+a_{31}b_{21} & a_{31}b_{12}+a_{32}b_{22} \end{pmatrix}$$

1.12.4 Square matrix:

A matrix having same number of rows and columns is called a square matrix.

ie $(A)_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & & & a_{nn} \end{pmatrix}$

a_{11}, a_{22}, a_{33}, a_{44}, …….a_{nn} are the diagonal elements of the square matrix A. The sum of the diagonal elements of a square matrix is called the Trace of that matrix ($T_r = a_{11}+a_{22}+.....a_{nn}$).

1.12.5 Diagonal matrix

If all the elements of a square matrix are zero except the diagonal elements the matrix is called as diagonal matrix.

$\begin{pmatrix} a_{11} & 0 & . & . & 0 \\ 0 & a_{22} & . & . & 0 \\ . & & & & \\ . & & & & \\ 0 & 0 & & & a_{nn} \end{pmatrix}$ is a diagonal matrix of order n.

And if in a diagonal matrix in which each diagonal element is unity then it is called an

identity or unit matrix. $\begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & & & & \\ . & & & & \\ 0 & 0 & & & 1 \end{pmatrix}$

1.12.6 Row or Column matrix

A matrix containing only one row or one column is called a vector.

A matrix of one row only of order 1 × n is called a row matrix. $[x]_{1\times n} = [a_{11}, a_{12}, a_{13}, \ldots\ldots a_{11}]$

A matrix of one column only of order m × 1 is called a column matrix $[x]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{bmatrix}$

Problem:

A= $\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ compute AB and BA and Show that AB ≠ BA

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$$

Hence AB ≠ BA.

1.12.7 Transpose of a matrix:

A matrix of order (n × m) is obtained by interchanging rows and columns of a matrix A of order (m × n) is called transpose of matrix A. It is denoted by A' or A^T or Ã .

And $(A^T)^T = A$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

1.12.8 Conjugate of a matrix:

If A is any matrix having complex numbers then the matrix obtained from A by replacing its each element by its conjugate complex number(ie., changing the sign of 'i' term) is called the conjugate of matrix A and is denoted by $\bar{A}$ or A^*. And $(A^*)^* = A$

Example: $A = \begin{bmatrix} 1+2i & 2 & i \\ 3 & 5i & 2-3i \end{bmatrix}$ Then $A^* = \begin{bmatrix} 1-2i & 2 & -i \\ 3 & -5i & 2+3i \end{bmatrix}$

1.12.9 Conjugate Transpose or Transpose Conjugate of a matrix :

Conjugate followed by Transpose or Transpose followed by conjugate is denoted by $A^{\dagger}$. ie., $A^{\dagger} = (A^{*})^{T} = (A^{T})^{*}$.

Example: $A = \begin{bmatrix} 1+2i & 5i & 2+7i \\ 2-i & 1 & -i \\ 0 & 4 & 3-7i \end{bmatrix}$ then $A^{\dagger} = (A^{*})^{T} = (A^{T})^{*} = \begin{bmatrix} 1-2i & 2+i & 0 \\ -5i & 1 & 4 \\ 2-7i & i & 3+7i \end{bmatrix}$

1.13 Determinant of a Matrix:

The determinant of a matrix is a special number that can be calculated from the elements of a square matrix. The determinant of a square matrix A is denoted by "det A" or | A |. The determinant helps us to find the inverse of a matrix. If the value of determinant of a matrix is zero ie.,| A | = 0, then the matrix is singular and if | A | ≠ 0 then the matrix is nonsingular.

The determinant value of a 2×2 matrix is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

= ad – bc

The determinant value of a 3×3 matrix is $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

= a (e i – f h) – b (d i – f g) + c (d h – e g)

Example:

$$|A| = \begin{vmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{vmatrix}$$

= 6 × (-2×7 – 5×8) – 1 × (4×7 – 5×2) + 1 × (4×8 – 2×2)

= 6 × (-54) – 1 × (18) + 1 × (36)

= – 306

1.13.1 Minors:

A minor is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix. These minors are labeled according to the row and column we deleted. The notation M_{ij} is used to stand for the minor of the element in row i and column j. So M_{21} would mean the minor for the element in row 2, column 1.

Consider the 3×3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

the minor a_{11} is obtained by deleting 1st row and 1st column of the determinant $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

the minor a_{12} is obtained by deleting 1st row and 2nd column of the determinant $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

the minor a_{13} is obtained by deleting 1st row and 3d column of the determinant $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly the minors of a21, a22, a23 and the minors of a31, a32, a33 are found as follows

$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix}$ and $\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Example:

Find the determinant value and matrix of minors for the given determinant $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$

Solution:

$$| A | = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$$

$= 1 \times (1\times2 - 3\times5) - 3 \times (4\times2 - 3\times2) + 2 \times (4\times5 - 1\times2)$

$= 1 \times (-13) - 3 \times (2) + 2 \times (18)$

$= 17$

To find matrix of minor,

Let the general matrix of minors for a 3 × 3 determinant is given below. Where C's represents the column number and R's represents the row numbers, whereas M's are the corresponding minors.

	C_1	C_2	C_3
R_1	M_{11}	M_{12}	M_{13}
R_2	M_{21}	M_{22}	M_{23}
R_3	M_{31}	M_{32}	M_{33}

Therefore the given determinant can be written in the general form

	C_1	C_2	C_3
R_1	1	3	2
R_2	4	1	3
R_3	2	5	2

To find each minor

.

	C_1	C_2	C_3
R_1	$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix}$ =2-15=-13	$\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix}$ =8-6=2	$\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$ =20-2=18
R_2	$\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix}$ =6-10=-4	$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$ =2-4=-2	$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ =5-6=-1
R_3	$\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$ =9-2=7	$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$ =3-8=-5	$\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}$ =1-12=-11

Then the matrix of minors is

	C_1	C_2	C_3
R_1	-13	2	18

R_2	-4	-2	-1
R_3	7	-5	-11

$$= \begin{pmatrix} -13 & 2 & 18 \\ -4 & -2 & -1 \\ 7 & -5 & -11 \end{pmatrix}$$

1.13.2 Cofactors of a determinant:

A cofactor for any element is either the minor or the opposite of the minor, depending on where the element is in the original determinant. If the row and column of the element add up to be an even number, then the cofactor is the same as the minor. If the row and column of the element add up to be an odd number, then the cofactor is the opposite of the minor.

The sign chart for a 3×3 determinant.

	C_1	C_2	C_3
R_1	+	-	+
R_2	-	+	-
R_3	+	-	+

The + does not mean positive and the - negative. The + means the same sign as the minor and the - means the opposite of the minor.

The matrix of cofactors is the matrix found by replacing each element of a matrix by its cofactor. This is the matrix of minors with the signs changed on the elements in the - positions.

	C_1	C_2	C_3
R_1	-13	-2	18
R_2	4	-2	1
R_3	7	5	-11

$$= \begin{pmatrix} -13 & -2 & 18 \\ 4 & -2 & 1 \\ 7 & 5 & -11 \end{pmatrix}$$

1.13.3 Adjoint of a matrix:

The adjoint of a matrix A is defined as the transpose of the matrix formed by the cofactors of elements of the determinant A. To transpose a matrix, interchange the rows and columns. That is, the rows become columns and the columns become rows. The adjoint of the above matrix of cofactors is given below.

		C_1	C_2	C_3
	R_1	-13	4	7
adj(A) =	R_2	-2	-2	5
	R_3	18	1	-11

$$= \begin{pmatrix} -13 & 4 & 7 \\ -2 & -2 & 5 \\ 18 & 1 & -11 \end{pmatrix}$$

<u>1.13.4 Inverse of a Matrix</u>:

To find inverse of a matrix A, the matrix must be nonsingular square matrix ie., | A | ≠ 0.

$$A^{-1} = \frac{adjA}{\det A}$$

Example:

Find the inverse of the given matrix $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$

Solution:

Given matrix is same as the above example; the same procedure is adapted to find $adjA$

From the above example we get, adj A = $\begin{bmatrix} -13 & 4 & 7 \\ -2 & -2 & 5 \\ 18 & 1 & -11 \end{bmatrix}$

And the determinant value is | A | = 17

$$A^{-1} = \frac{\begin{bmatrix} -13 & 4 & 7 \\ -2 & -2 & 5 \\ 18 & 1 & -11 \end{bmatrix}}{17}$$

$$A^{-1} = \begin{bmatrix} -13/17 & 4/17 & 7/17 \\ -2/17 & -2/17 & 5/17 \\ 18/17 & 1/17 & -11/17 \end{bmatrix}$$

1.14 Orthogonal matrix:

A square matrix A is said to be orthogonal when it satisfies the relations $A^T A = AA^T = I$. Where A^T is the transpose of A and I is the unit matrix.

Example:

Show that the following matrix is orthogonal $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, Then $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Therefore $AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence the given matrix is Orthogonal.

1.15 Unitary Matrices:

A square matrix A is said to be Unitary when it satisfies the relations $A^\dagger A = AA^\dagger = I$. Where $A^\dagger$ is the conjugate transpose of A and I is the unit matrix.

Example: Show that the following matrix is unitary $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Let $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, Then $A^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Therefore $AA^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Hence the given matrix is Unitary.

1.16 Eigen Values and Eigen Vectors:

For a square matrix A of order n, the number λ is an eigenvalue if and only if there exists a non-zero vector X such that

$$A\,X = \lambda\,X$$

Using the matrix multiplication properties, we obtain $(A - \lambda I_n)\,X = 0$ This is a linear system for which the matrix coefficient is $A - \lambda I_n$. We also know that this system has one solution if and only if the matrix coefficient is invertible, i.e. $det(A - \lambda I_n) \neq 0$. Since the zero-vector is a solution and X is not the zero vector, then we have $det(A - \lambda In) = 0$. In general, for a square matrix A of order n, the equation

$$det(A - \lambda I_n) = 0 \;\; ie., |A - \lambda I_n| = 0$$

Will give the eigenvalues of A. This equation is called the characteristic equation or characteristic polynomial of A. It is a polynomial function in of degree n. So we know that this equation will not have more than n roots or solutions. So a square matrix A of order n will not have more than n eigenvalues.

Example: Find the eigen values eigen vector of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and the characteristic equation is $|A - \lambda I| = 0$

ie., $|A - \lambda I| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = 0$

$(1 - \lambda)\{(1 - \lambda)^2 - 1\} = 0 \Rightarrow (1 - \lambda)\{1 + \lambda^2 - 2\lambda - 1\} = 0 \Rightarrow (1 - \lambda)(\lambda - 2)\lambda = 0$

ie., $\lambda = 0, 1, 2$

The eigen values of the matrix A are $0, 1, 2$

And the eigen value equation is $(A - \lambda I)\,X = 0$

Case 1, λ =0, the eigen value equation is $\begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ we get $\begin{matrix}x_1=0\\x_2+x_3=0\\x_2+x_3=0\end{matrix}$

Solving these equations we get $x_1=0; x_2=-x_3$, X_1 = $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\-x_2\end{pmatrix} = \begin{pmatrix}0\\k\\-k\end{pmatrix}$

To normalize the eigen vector it must be equated to unity $|X_1|$ =1, ie., $\sqrt{0^2+\ k^2+\ (-k)^2}=1;\ \sqrt{2k^2}=1$ therefore $k=\frac{1}{\sqrt{2}}$. $\therefore$ the normalized eigen vector of matrix A for λ =0 is $\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}$

Case 2, $\lambda = 1$, the eigen value equation is $\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ we get $\begin{matrix}x_3=0\\x_2=0\end{matrix}$

So that $X_2 = \{x_1,\ x_2,\ x_3\} = \{1,\ 0,\ 0\}$ is the suitable eigen vector and is normalized.

Case 3, $\lambda = 2$, the eigen value equation is $\begin{pmatrix}-1&0&0\\0&-1&1\\0&1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ we get

$$\begin{matrix}-x_1=0\\-x_2+x_3=0\\x_2-x_3=0\end{matrix}$$

Solving these equations we get $x_1=0;\quad x_2=x_3$ X_3 = $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_2\end{pmatrix} = \begin{pmatrix}0\\k\\k\end{pmatrix}$

$\therefore$ the normalized the eigen vector is $\sqrt{0^2+\ k^2+\ k^2}=1;\quad \sqrt{2k^2}=1$ therefore $k=\frac{1}{\sqrt{2}}$.

$\therefore$ the normalized eigen vector of matrix A for λ =2 is $\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}$

1.17 Diagonalization:

To reduce a given square matrix A to diagonal form, evaluate the characteristic roots(or eigen values) λ_1, , λ_2, …… λ_n from the characteristic equation of the matrix A. Then the required diagonal matrix D of A can be obtained as the following method.

$$D = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Example1: Diagonalize the matrix $\begin{bmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$ and the Characteristic equation is $|A - \lambda I| = \begin{vmatrix} \frac{4}{3}-\lambda & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3}-\lambda \end{vmatrix} = 0$

$$\left(\frac{4}{3}-\lambda\right)\left(\frac{5}{3}-\lambda\right) - \frac{\sqrt{2}}{3}.\frac{\sqrt{2}}{3} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0 \Longrightarrow (\lambda - 1)(\lambda - 2) = 0$$

$$\therefore \lambda = 1 \ \ and \quad \lambda = 2$$

Then the required diagonal matrix is $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Example2: Diagonalize the matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and the characteristic equation $|A-\lambda I| = \begin{bmatrix} \cos\theta-\lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

$$(\cos\theta-\lambda)\{(\cos\theta-\lambda)(1-\lambda)-0\} + \sin\theta\,\{\sin\theta\,(1-\lambda)-0\} = 0$$
$$(1-\lambda)\left\{(\cos\theta-\lambda)^2+\sin^2\theta\right\} = 0$$
$$(1-\lambda)\left\{\cos^2\theta-2\lambda\cos\theta+\lambda^2+\sin^2\theta\right\} = 0$$
$$(1-\lambda)\left\{\lambda^2-2\lambda\cos\theta+1\right\} = 0$$

The roots are $\lambda = 1 \; and \; \lambda = \dfrac{2\cos\theta \pm \sqrt{(4\cos^2\theta-4)}}{2}$

ie., $\lambda = 1 \; and \; \lambda = \cos\theta \pm \sin\theta$; $\lambda = 1 \; and \; \lambda = e^{\pm i\theta}$ then the eigen values $\lambda_1 = 1 \;\; \lambda_2 = e^{i\theta} \;\; \lambda_3 = e^{-i\theta}$

and the diagonal matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{bmatrix}$

1.18 Cayley – Hamilton Theorem:

Every square matrix satisfies its own characteristics equation. For a square matrix A of order n, the characteristic polynomial is $|A-\lambda I| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots\cdots\cdots + a_n\lambda^n$

Then the matrix equation $a_0 I + a_1 X + a_2 X^2 + \cdots\cdots\cdots + a_n X^n = 0$ is satisfied by $X = A$.

Proof:

The characteristic polynomial is $|A-\lambda I| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots\cdots\cdots + a_n\lambda^n$

The characteristic equation of A is $|A-\lambda I| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots\cdots\cdots + a_n\lambda^n = 0$

Then the matrix equation is $a_0 I + a_1 X + a_2 X^2 + \cdots\cdots\cdots + a_n X^n = 0$

If the matrix equation is satisfied by A, then $a_0 I + a_1 A + a_2 A^2 + \cdots\cdots\cdots + a_n A^n = 0$

Since each element of the characteristic matrix $(A-\lambda I)$ is an ordinary polynomial of degree n then the cofactor of every element of $(A-\lambda I)$ is an ordinary polynomial of degree (n-1). Therefore each element of $B = adj(A-\lambda I)$ is an ordinary polynomial of degree (n-1).

$\therefore$ We can write $B = adj(A-\lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \cdots\cdots\cdots\cdots + B_{n-1}\lambda^{n-1}$ where $B_0, B_1, B_2, \cdots\cdots\cdots B_{n-1}$ are all square matrices of the same order n whose elements are polynomials in the elements of the square matrix A. We have, $(A-\lambda I)\,adj(A-\lambda I) = |A-\lambda I|\,I$

$(A-\lambda\, I)(B_0 + B_1\lambda + B_2\lambda^2 + \cdots\cdots + B_{n-1}\lambda^{n-1}) = (a_0 + a_1\lambda + a_2\lambda^2 + \cdots\cdots + a_n\lambda^n)I$

Comparing the coefficient of like powers of λ on both sides we get,

$AB_0 = a_0 I$

$AB_1 - B_0 = a_1 I$

$AB_2 - B_1 = a_2 I$

………………

$AB_{n-1} - B_{n-2} = a_{n-1} I$

$-B_{n-1} = a_n I$

Now pre multiplying these equations by $I, A, A^2, \cdots\cdots\cdots A^n$ and then adding we get

$0 \;=\; a_0 I + a_1 A + a_2 A^2 + \cdots\cdots\cdots\cdots + a_n A^n$ This proves the theorem.

Note: From This equation we can find the inverse of the square matrix A.

ie., $-a_0 I \;= a_1 A + a_2 A^2 + \cdots\cdots\cdots\cdots + a_n A^n$

$$I \;= -\left(\frac{a_1}{a_0}A + \frac{a_2}{a_0}A^2 + \cdots\cdots\cdots\cdots + \frac{a_n}{a_0}A^n\right)$$

Then pre multiplying the equation by A^{-1} on both sides we get the inverse of the matrix A.

$$A^{-1} \;= -\left(\frac{a_1}{a_0}I + \frac{a_2}{a_0}A + \cdots\cdots\cdots\cdots + \frac{a_n}{a_0}A^{n-1}\right)$$

Example:

Find the characteristic equation of the matrix A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A. Hence find the inverse of A.

Solution :

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$; $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

This is the required characteristic equation of A. If the characteristic equation is satisfied by A , we must have, $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

So that the equation $A^3 - 6A^2 + 9A - 4I = 0$ becomes

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ This verifies the Cayley – Hamilton theorem.

To find A^{-1}, $A^3-6A^2+9A-4I=0$

$4I=A^3-6A^2+9A$

$4I=A\left(A^2-6A+9I\right)$

$\frac{I}{A}=\frac{1}{4}\left(A^2-6A+9I\right)$

$A^{-1}=\frac{1}{4}\left(A^2-6A+9I\right)$ = $\frac{1}{4}\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$ − $\frac{6}{4}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ + $\frac{9}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

= $\begin{bmatrix}\frac{3}{4} & \frac{1}{4} & \frac{-1}{4}\\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4}\\ \frac{-1}{4} & \frac{1}{4} & \frac{3}{4}\end{bmatrix}$

UNIT II : FUNCTIONS AND POLYNOMIALS

Beta , Gamma functions – Dirac delta function and its properties – Green's function – Bessel differential equation – Generating function for $J_n(x)$ – Recurrence relation for $J_n(x)$ – Legendre differential equation – Generating function for $P_n(x)$ – Recurrence relation for $P_n(x)$- Hermite differential equation – Generating function for $H_n(x)$ – Recurrence relation for $H_n(x)$

2.1 Beta Function:

The beta function of m, n written $\beta(m,n)$ is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx \qquad (2.1)$$

Which converges when$m$ and n positive integers, $\beta(m,n)$ = $\beta(n,m)$ $\therefore$ β function is symmetrical in m and n

ie., put $x = 1-y$ and $dx = -dy$ in equation (2.1) we get

$$\beta(m,n) = -\int_0^1 (1-y)^{m-1}\,y^{n-1}\,dy = \int_1^0 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

Thus we prove that $\beta(m,n) = \beta(n,m)$ (2.2)

Another expression of $\beta(m,n)$ can be obtained by substituting

$x = sin^2\theta \Longrightarrow dx = 2sin\theta\ cos\theta\ d\theta$

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1}\,(cos^2\theta)^{n-1}\,2sin\theta\ cos\theta\ d\theta$$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2n-1}\theta\ \ d\theta \qquad (2.3)$$

2.1.1 Other Form of Beta Function:

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx Let\ us\ put\ x = \frac{y}{(1+y)}$$

$$so\ that\ \ dx = \frac{(1+y)-y}{(1+y)^2}\ ;\quad dy = \frac{1}{(1+y)^2}\,dy\ \ and\ \ 1-x = \frac{1}{(1+y)}\ then$$

$$\beta(m,n) = \int_0^1 \frac{y^{m-1}}{(1+y)^{m-1}}\frac{1}{(1+y)^{n-1}}\frac{1}{(1+y)^2}dx = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \qquad (2.4)$$

Since $\beta(m,n)$= $\beta(n,m)$, we have

$$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy \qquad (2.5)$$

(2.5) is one more form of beta function.

2.1.2 Evaluation of Beta Function

By definition,

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$$

On integrating by parts and keeping $(1-x)^{n-1}$ as the first function, we have

$$\beta(m,n) = \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 + \int_0^1 (n-1)(1-x)^{n-2}\frac{x^m}{m}\;dx$$

$$= \frac{n-1}{m}\int_0^1 (1-x)^{n-2}\,x^m\;dx$$

Integrating by parts again , we get

$$\beta(m,n) = \frac{(n-1)(n-2)}{m(m+1)}\int_0^1 (1-x)^{n-3}\,x^{m+1}dx$$

On continuing this process with n is positive integer, we get

$$\beta(m,n) = \frac{(n-1)(n-2)\cdots\cdots 2.1}{m(m+1)\cdots\cdots(m+n-2)}\int_0^1 x^{m+n-2}\,dx$$

$$= \frac{(n-1)(n-2)\cdots\cdots 2.1}{m(m+1)\cdots\cdots(m+n-2)}\left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1$$

$$= \frac{(n-1)!}{m(m+1)\cdots\cdots(m+n-2)(m+n-1)}$$

Again if m is a positive integer, then

$$\beta(m,n) = \frac{(n-1)!\,(m-1)!}{(m+n-1)!} \qquad (2.6)$$

By the definition of Beta function using equation (2.1), $\beta(1,1) = 1$

By the definition of Beta function using equation (2.3), $\beta\left(\frac{1}{2},\,\frac{1}{2}\right) = \pi$

Beta function is also known as '*Euler's integral of the first kind*'**.**

Problems:

1. Find the values of (i) $\beta(m,n+1)$ (ii) $\beta(m,1)$ (iii) $\beta(m,2)$

We have $\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$

$$\beta(m,n+1) = \int_0^1 x^{m-1}\,(1-x)^n\,dx \qquad m > 0, \qquad n > -1$$

If $n = 0, \beta(m,1) = \int_0^1 x^{m-1}\,(1-x)^0\,dx$

$$= \int_0^1 x^{m-1}\;dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}$$

Tf$n = 1, \beta(m,2) = \int_0^1 x^{m-1}\,(1-x)\,dx$

$$= \left[\frac{x^m}{m}\,(1-x)\right]_0^1 + \frac{1}{m}\int_0^1 x^m\;dx$$

$$= 0 + \left[\frac{1}{m}\frac{x^{m+1}}{m+1}\right]_0^1$$

$$= \frac{1}{m(m+1)}$$

Continuing Integration, we get

$$\beta(m, n+1) = \frac{n(n+1)\cdots\cdots 1}{m(m+1)\cdots(m+n-1)}\int_0^1 x^{m+n-1}\;dx$$

$$= \frac{1\cdot 2\cdot 3\cdots\cdots n}{m(m+1)\cdots(m+n)}$$

2.2 Gamma Function

A Gamma function $\Gamma(\mathrm{n})$ with $n > 0$ is defined as

$$\Gamma(\mathrm{n}) = \int_0^\infty e^{-x}\,x^{n-1}dx \qquad (2.7)$$

Recurrence Relation:

(i) By definition we have

$$\Gamma(\mathrm{n}) = \int_0^\infty e^{-x}\,x^{n-1}dx$$

Integrating by parts and keeping x^{n-1} as the first function, we get

$$\Gamma(\mathrm{n}) = \left[-x^{m-1}e^{-x}\right]_0^\infty + \int_0^\infty (n-1)\,x^{n-2}e^{-x}\;dx$$

$$\Gamma(\mathrm{n}) = (n-1)\int_0^\infty x^{n-2}\,e^{-x}\;dx$$

$$= (n-1)\int_0^\infty x^{n-2}\,e^{-x}\;dx$$

$$\Gamma(\mathrm{n}) \quad = (n-1)\Gamma(\mathrm{n}-1) \qquad (2.8)$$

$$= (n-1)(n-2)\Gamma(\mathrm{n}-2) = \cdots\cdots$$

$$= (n-1)(n-2)\cdots\cdots 2\cdot 1 = (\mathrm{n}-1)!$$

$$\therefore \Gamma(n) = (n-1)! \qquad (2.9)$$

(ii) By definition we have

$$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n dx$$

Integrating by parts, we get $= [-x^n e^{-x}]_0^\infty + n\int_0^\infty e^{-x}\, x^{n-1} dx$

$$= 0 + n\,\Gamma(n)$$

$$\therefore \Gamma(n+1) = n\Gamma(n) \qquad (2.10)$$

Values of $\Gamma(n)$ in terms of factorial

$\Gamma(1+1) = \Gamma(2) = 1 \times \Gamma(1) = 1!$

$\Gamma(2+1) = \Gamma(3) = 2 \times \Gamma(2) = 2 \times 1 = 2!$

$\Gamma(3+1) = \Gamma(4) = 3 \times \Gamma(3) = 3 \times 2! = 3!$

$\cdots\cdots = \cdots\cdots$

$$\Gamma(n+1) = n! \qquad (2.11)$$

Therefore Gamma function is considered as factorial function. And also known as '*Euler's integral of second kind*'.

When $n = 0$, the relation (2.10) defines $0! = \Gamma(0+1) = \Gamma(1) = 1$ (2.12)

Problems:

1. Find the value of $\Gamma\left(\frac{1}{2}\right)$

We have $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}\, x^{\frac{1}{2}-1} dx = \int_0^\infty e^{-x}\, x^{-\frac{1}{2}}\, dx$$

Let $x = y^2$ and $dx = 2y\, dy$

$$= \int_0^\infty e^{-y^2} (y^2)^{-\frac{1}{2}}\, 2\, ydy$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\, dy$$

$Similarly, we\ can\ write$ $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx$

$Then$ $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-y^2}\, dy \int_0^\infty e^{-x^2}\, dx$

$$=4\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Using polar coordinates (r,θ) *so that* $x^2+y^2 = r;\ dx\,dy = r\,dr\,d\theta$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_{x=0}^{\frac{\pi}{2}}\int_{y=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$=4\tfrac{\pi}{2}\int_0^{\infty} e^{-r^2}\, r\,dr$$

$$=2\pi\left[\left(-\tfrac{1}{2}\right)e^{-r^2}\right]_0^{\infty} = \pi$$

$$\therefore\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2.3 Relation between Beta and Gamma functions:

From the definition we have

$$\Gamma(\mathrm{m}) = \int_0^{\infty} e^{-t}\, x^{m-1} dt$$

Put $t = x^2 \Longrightarrow dt = 2x\,dx$

$$\Gamma(\mathrm{m}) = 2\int_0^{\infty} e^{-x^2}\, x^{2m-1} dx$$

Similarly,

$$\Gamma(\mathrm{n}) = 2\int_0^{\infty} e^{-y^2}\, y^{2n-1} dy$$

$$\therefore\ \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, x^{2m-1} y^{2n-1} dx\,dy$$

By Polar coordinates (r,θ) *we have* $x = r\cos\theta$, $y = r\sin\theta$ *and* $dx\,dy = r\,d\theta\,dr$

$$\therefore\ \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2}\, r^{2(m+n)-1} cos^{2m-1}\theta\, sin^{2n-1}\theta\, d\theta\, dr$$

$$= 2\int_0^{\frac{\pi}{2}} cos^{2m-1}\theta\, sin^{2n-1}\theta\, d\theta \times 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr$$

By definition first integral is $\beta(m,n)$ and the second integral is $\Gamma(\mathrm{m}+\mathrm{n})$

ie., $\Gamma(\mathrm{m})\Gamma(\mathrm{n}) = \beta(m,n) \times \Gamma(\mathrm{m}+\mathrm{n})$

$$\therefore\quad \beta(m,n) = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{m}+\mathrm{n})}$$

2.4 Dirac delta function :

The Dirac delta function is an infinitely thin spike. In one dimensional, it is expressed as follow

$$\delta(x) = 0 \ \ when\ x \neq 0; \int_{-\infty}^{\infty} \delta(x)dx = 1; \int_{-\infty}^{\infty} \delta(x)\, f(x)dx = f(0)$$

The function $f(x)$ is assumed to be continuous around $x = 0$.

2.4.1 Properties of Dirac delta function in one dimension:

(i) $\delta(x) = \delta(-x)$

(ii) $x\, \delta(x) = 0$

(iii) $\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$

(iv) $\int_{-\infty}^{\infty} f(x)\delta(x+a)\, dx = f(-a)$

(v) $\int_{-\infty}^{\infty} \delta(x-a)\, \delta(x-b)dx = \delta(a-b)$

In three dimensional space, it is expressed as

$$\delta(\vec{r}) = 0\ when\ \vec{r} \neq 0; \int_{-\infty}^{\infty} \delta(\vec{r})\, d^3\vec{r} = 1; \int_{-\infty}^{\infty} \delta(\vec{r})\, f(\vec{r})d^3\vec{r} = f(0)$$

2.4.2 Properties of Dirac delta function in three dimensions:

(i) $\delta(\vec{r_2} - \vec{r_1}) = \delta(\vec{r_1} - \vec{r_2})$

(ii) $\delta(\vec{r_1} - \vec{r_2}) = 0$ when $\vec{r_1} \neq \vec{r_2}$

(iii) $\int_{-\infty}^{\infty} \delta(\vec{r} - \vec{r_1})\ \delta(\vec{r} - \vec{r_2})\, d^3\vec{r} = \delta(\vec{r_1} - \vec{r_2})$

2.5 Green Function:

To understand the Green's function, consider the differential equation,

$$L\, u(x) = f\ (x) \qquad (2.13)$$

Where L is an ordinary linear differential operator, $f\ (x)$ is a known function while $u(x)$ is an unknown function. To solve above equation, one method is to find the inverse operator L^{-1} in the form of an integral operator with a kernel $G(x, \xi)$such that,

$$u(x) = L^{-1} f\ (x) = \int G(x, \xi)\, f\ (\xi) d\xi \qquad (2.14)$$

The kernel of this integral operator is called Green's function for the differential operator. Thus the solution to the non-homogeneous differential equation (1) can be written down, once the Green's function for the problem is known. For this reason, the Green's function is also sometimes called the fundamental solution associated to the operator L.

2.5.1 Green function method in electrostatics:

For the continuous point charge distribution with charge density $(\vec{r})$, then potential can be written as

$$\psi(0) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r})}{\vec{r}}\, d^3\,\vec{r}$$

The potential is measured at the origin of the coordinate system, the potential at $\vec{r}= \vec{r_1}$ due to charges at $\vec{r}= \vec{r}_2$ is given as

$$\psi(\vec{r}_1) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|}\, d^3\vec{r}_2 \qquad (2.15)$$

The potential using Green function is written as

$$\psi(\vec{r}_1) = \frac{1}{\varepsilon_0}\int G(\vec{r}_1,\vec{r}_2)\rho(\vec{r}_2) d^3\vec{r}_2 \qquad (2.16)$$

Comparing equation (3) and (4) we get,

$$G(\vec{r}_1,\vec{r}_2) = \frac{1}{4\pi|\vec{r}_2 - \vec{r}_1|}$$

2.6 Bessel Function

Bessel differential equation is

$$x^2\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad (2.17)$$

Where n is an integer or a half integer. Solution for this equation is known as Bessel function.

To get singular points and solutions for the equation (2.17), modify the equation as

$$x^2\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + x^2\left(1 - \frac{n^2}{x^2}\right)y = 0$$

Dividing by x^2, we get

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

$$Let \quad P(x) = \frac{1}{x} \quad and \quad Q(x) = 1 - \frac{n^2}{x^2}$$

There is a singular point at $x = 0$. The singular point is regular as

$$(x-0)\, P(x) \qquad and \qquad (x-0)^2\, Q(x)$$

Both are finite at the point $x = 0$. ie., the singular point is irregular. By Fuchs theorem, Bessel equation has a series solution. They are given as,

$$y = \sum_{m=0}^{\infty} a_m\, x^{m+k} a_0 \neq 0 \qquad (2.18)$$

On differentiating with respect to x, we get

$$y' = \sum_{m=0}^{\infty} a_m\, (m+k) x^{m+k-1} \qquad (2.19)$$

and

$$y'' = \sum_{m=0}^{\infty} a_m\, (m+k)(m+k-1) x^{m+k-2} \qquad (2.20)$$

Substituting equations (2.18), (2.19) and (2.20) in (2.17) we get

$$\sum_{m=0}^{\infty} a_m\, (m+k)(m+k-1) x^{m+k} + \sum_{m=0}^{\infty} a_m\, (m+k) x^{m+k} + (x^2 - n^2) \sum_{m=0}^{\infty} a_m\, x^{m+k} = 0$$

$$\sum_{m=0}^{\infty} a_m\, [(m+k)(m+k-1) + (m+k) - n^2] x^{m+k} + \sum_{m=0}^{\infty} a_m\, x^{m+k+2} = 0$$

$$\sum_{m=0}^{\infty} a_m\, [(m+k)^2 - n^2] x^{m+k} + \sum_{m=0}^{\infty} a_m\, x^{m+k+2} = 0 \qquad (2.21)$$

Equation (2.21) is a polynomial equation. Equating the coefficient of the lowest power of x to zero, we get

$a_0\,(k^2 - n^2) = 0$, since $a_0 \neq 0$, $\therefore (k^2 - n^2) = 0$, $k^2 = n^2$ and $k = \pm n$

Equating the coefficient x^{k+1} to zero, we get $a_1\,[(1+k)^2 - n^2] = 0$

For $k = \pm n$, we have $[(1+k)^2 - n^2] \neq 0$ and $\therefore\ a_1 = 0$

Further equating the coefficient of x^{k+r} to zero, we get $a_r\,[(r+k)^2-n^2]+a_{r-2}=0$

$$\therefore\; a_r = -\frac{a_{r-2}}{(r+k+n)(r+k-n)} \qquad (2.22)$$

Since $a_1 = 0$, equation (6) gives $a_3 = a_5 = a_7 = \cdots = 0$

But $a_0 \neq 0$, we have non –zero values for $a_2, a_4, a_6, \cdots$

Case 1 when $k = n$,

$$a_r = -\frac{a_{r-2}}{(r+2n)r}$$

$$a_2 = -\frac{a_0}{(2+2n)2} = -\frac{a_0}{2\cdot 2(n+1)}$$

$$a_4 = -\frac{a_2}{(4+2n)4} = -\frac{a_2}{2\cdot 4(n+2)} = \frac{a_0}{2\cdot 2(n+1)2\cdot 4(n+2)} = \frac{a_0}{2^4\cdot 1\cdot 2(n+1)(n+2)}$$

$$a_6 = -\frac{a_4}{(6+2n)6} = -\frac{a_4}{2\cdot 6(n+3)} = -\frac{a_0}{2^6\cdot 1\cdot 2\cdot 3(n+1)(n+2)(n+3)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{2r} = (-1)^r\frac{a_0}{2^{2r}\cdot 1\cdot 2\cdot 3\cdots r(n+1)(n+2)\cdots(n+r)} \quad (2.23)$$

If we take $$a_0 = \frac{1}{2^n\Gamma(n+1)} \qquad (2.24)$$

Put (2.24) in (2.23), we get

$$a_{2r} = (-1)^r\frac{1}{2^{n+2r}\cdot r!\,\Gamma(n+r+1)} \qquad (2.25)$$

Since $a_1 = a_3 = a_5 = a_7 = \cdots = 0$, equation (2) can be written as

$$y = \sum_{r=0}^{\infty} a_{2r}\,x^{2r+n} a_0 \neq 0 \qquad (2.26)$$

Substitute (2.25) in (2.26) we get the Bessel function of first kind of order n, and is denoted by $J_n(x)$

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{n+2r}r!\,\Gamma(n+r+1)} = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \qquad (2.27)$$

Case 2 when$k = -n$, ie replacing n $by - n$ in equation (2.27)

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{2r-n}$$

Since the argument of Gamma function must be greater than zero, we have

$$J_{-n}(x)=\sum_{r=n}^{\infty}\frac{(-1)^r}{r!\,\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{2r-n} \qquad (2.28)$$

Put $r = s+n$ in equation (2.28), we get

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^{s+n}}{(n+s)!\,\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s+n} = (-1)^n\sum_{r=0}^{\infty}\frac{(-1)^s}{\Gamma(n+s+1)s!}\left(\frac{x}{2}\right)^{2s+n}$$

$$= (-1)^n J_n(x)$$

Thus we have $J_{-n}(x) = (-1)^n J_n(x)$

Therefore the linear combination of $J_n(x)$ *and* $J_{-n}(x)$ is the solution of Bessel equation

$$y = A\,J_n(x)+BJ_{-n}(x) = A\,J_n(x)+B\,(-1)^n J_n(x) = [A+B(-1)^n]J_n(x) = C\,J_n(x)$$

C is a constant therefore $J_n(x)$ is the general solution of Bessel differential equation (2.17).

2.6.1 Special cases for Bessel function $J_n(x)$

We have $J_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,r!}\left(\frac{x}{2}\right)^{2r}$

$$=1-\left(\frac{x}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4-\frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6+\cdots$$

Then $J_0(0) = 1$, we have

$$J_1(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(r+1)!}\left(\frac{x}{2}\right)^{2r+1}$$

$$=\frac{x}{2}-\frac{1}{2!}\left(\frac{x}{2}\right)^3+\frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5-\frac{1}{(3!)^2}\left(\frac{x}{2}\right)^7+\cdots$$

Thus $J_1(0)=0$, similarly $J_n(0)=0$ when $n \neq 0$ie.,$J_n(0)=\delta_{0n}$ where δ_{0n} is Kronecker delta function which has value one if $n=0$ otherwise zero if $n \neq 0$

$$J_{\frac{1}{2}}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(r+\frac{3}{2})}\left(\frac{x}{2}\right)^{2r+\frac{1}{2}}$$

$$= \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}} - \frac{1}{\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^{\frac{5}{2}} + \frac{1}{2!\,\Gamma(\frac{7}{2})}\left(\frac{x}{2}\right)^{\frac{9}{2}} - \cdots$$

$$= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x$$

$$J_{-\frac{1}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+\frac{1}{2})}\left(\frac{x}{2}\right)^{2r-\frac{1}{2}}$$

$$= \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{-\frac{1}{2}} - \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{3}{2}} + \frac{1}{2!\,\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^{\frac{7}{2}} - \cdots$$

$$= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x$$

2.6.2 Generating function for $J_n(x)$

The function $e^{x(t-\frac{1}{t})/2}$ is known as generating function for the Bessel function of first kind because $J_n(x)$ is coefficient of t^n in the expansion of this function. ie.,

$$e^{x(t-\frac{1}{t})/2} = \sum J_n(x) t^n \qquad (2.29)$$

Proof:

$$We\ know\ that\ e^{xt/2} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r r!} \qquad (2.30)$$

and

$$e^{-x/2t} = \sum_{s=0}^{\infty} \frac{(-1)^s x^s t^{-s}}{2^s s!} \qquad (2.31)$$

product of equation (2.30) and (2.31) is

$$e^{x(t-\frac{1}{t})/2} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r r!} \sum_{s=0}^{\infty} \frac{(-1)^s x^s t^{-s}}{2^s s!} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s x^{r+s} t^{r-s}}{2^{r+s} r!\, s!} \qquad (2.32)$$

Put $r = n + s$ in equation (2.32) we get

$$e^{x(t-\frac{1}{t})/2} = \sum_{s=0}^{\infty}\sum_{n=-s}^{\infty}\frac{(-1)^s x^{n+2s} t^n}{2^{n+2s}(n+s)!\,s!}$$

$$= \sum_{n=-s}^{\infty}\left[\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(\mathrm{n+s+1})}\left(\frac{x}{2}\right)^{n+2s}\right]t^n$$

$$= \sum_{n=-s}^{\infty} J_n(x)t^n \qquad (2.33)$$

Put $n + r = s$ in equation (2.32) we get

$$e^{x(t-\frac{1}{t})/2} = \sum_{r=0}^{\infty}\sum_{n=-r}^{\infty}\frac{(-1)^{n+r} x^{n+2r} t^{-n}}{2^{n+2r}(n+r)!\,r!}$$

$$= \sum_{n=-r}^{\infty}(-1)^n\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(\mathrm{n+r+1})}\left(\frac{x}{2}\right)^{n+2r}\right]t^{-n}$$

$$= \sum_{n=-r}^{\infty}(-1)^n J_n(x)t^{-n} \quad = \sum_{n=-r}^{\infty} J_{-n}(x)t^{-n} \qquad (2.34)$$

Replacing –n by n in equation (2.34) we get

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=r}^{\infty} J_n(x)t^n \qquad (2.35)$$

This proves that in the expansion of the generating function $e^{x(t-\frac{1}{t})/2}$,the coefficient of t^n is $J_n(x)$

2.6.3 Recurrence relations for $J_n(x)$

Relations among various orders of the Bessel function are known as recurrence relations. We have

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=r}^{\infty} J_n(x)t^n \qquad (2.36)$$

(i) Differentiating (2.36) with respect to t, we get

$$e^{x(t-\frac{1}{t})/2}\left[\frac{x}{2}\left(1+\frac{1}{t^2}\right)\right] = \sum_{n=r}^{\infty} J_n(x)n\, t^{n-1} \qquad (2.37)$$

Then (2.36) in (2.37) we get,

$$\left[\frac{x}{2}\left(1+\frac{1}{t^2}\right)\right]\sum J_n(x)t^n = \sum_{n=r}^{\infty} J_n(x)n\, t^{n-1} \qquad (2.38)$$

Equating the coefficient of t^{m-1} on both sides of (2.38) we get

$$\frac{x}{2}J_{m-1}(x) + \frac{x}{2}\, J_{m+1}(x) = m\, J_m(x)$$

$$\boldsymbol{J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x}J_m(x)} \qquad (2.39)$$

(ii) Differentiating equation (2.36) with respect to x, we get

$$e^{x(t-\frac{1}{t})/2}\left[\frac{1}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=r}^{\infty} J'_n(x)\, t^n \qquad (2.40)$$

Then (2.36) in (2.40) we get,

$$\left[\frac{1}{2}\left(1-\frac{1}{t}\right)\right]\sum J_n(x)t^n = \sum_{n=r}^{\infty} J_n(x)\, t^n \qquad (2.41)$$

Equating the coefficient of t^m on both sides of (2.41) we get

$$\frac{1}{2}J_{m-1}(x) + \frac{1}{2}J_{m+1}(x) = J'_m(x)$$

$$J_{m-1}(x) - J_{m+1}(x) = 2\, J'_m(x) \qquad (2.42)$$

(iii) Adding Equations (2.39) and (2.42) we get

$$2J_{m-1}(x) = \frac{2m}{x}J_m(x) + 2J'_m(x)$$

$$xJ'_m(x) = x\, J_{m-1}(x) - m\, J_m(x) \qquad (2.43)$$

(iv) Subtracting equation (2.42) from (2.39), we get

$$2J_{m+1}(x) = \frac{2m}{x}J_m(x) - 2J'_m(x)$$

$$xJ'_m(x) = m\,J_m(x) - xJ_{m+1}(x) \qquad (2.44)$$

(v) Put m=0 in (2.44) we get $xJ'_0(x) = -xJ_1(x) \quad or \quad J'_0(x) = J_1(x)$

(vi) Put $m = \frac{1}{2}$ in equation (2.39), we get

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$

$$J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

On using the values of $J_{\frac{1}{2}}(x)$and $J_{-\frac{1}{2}}(x)$, we get

$$J_{\frac{3}{2}}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

(vii) Put $m = -\frac{1}{2}$ in equation (2.39) we get

$$J_{-\frac{3}{2}}(x) + J_{\frac{1}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x)$$

$$J_{-\frac{3}{2}}(x) + J_{\frac{1}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

On using the values of $J_{\frac{1}{2}}(x)$and $J_{-\frac{1}{2}}(x)$, we get

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x = \sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

2.7 Legendre Differential Equation

Legendre differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + n(n+1)y = 0 \qquad (2.45)$$

Where n is a positive integer. Solution for this equation is known as Legendre function.

Rewrite the equation as

$$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{n(n+1)}{(1-x^2)}y = 0$$

Let $P(x) = -\frac{2x}{1-x^2}$ and $P(x) = \frac{n(n-1)}{1-x^2}$

The singular points are $x = -1$ and $x = 1$. The singular points are regular as $(x+1)P(x)$ and $(x+1)^2Q(x)$ both are finite at $x = -1$ and $(x-1)P(x)$ and $(x-1)^2Q(x)$ both are finite at $x = 1$. By Fluchs theorem, Legendre differential equation has a series solution. ie.,

$$y = \sum_{m=0}^{\infty} a_m\, x^{k-m} a_0 \neq 0 \qquad (2.46)$$

Differentiating (2.46) with respect to x, we get

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} a_m\,(k-m)x^{k-m-1} \qquad (2.47)$$

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} a_m\,(k-m)(k-m-1)\,x^{k-m-2} \qquad (2.48)$$

Using equations (2.46), (2.47) and (2.48) in (2.45) we get

$$(1-x^2)\sum_{m=0}^{\infty} a_m\,(k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} a_m\,(k-m)x^{k-m-1} + n(n+1)\sum_{m=0}^{\infty} a_m\,x^{k-m} = 0$$

$$\sum_{m=0}^{\infty} a_m\,(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} a_m\,[(k-m)(k-m+1) - n(n+1)]x^{k-m} = 0 \qquad (2.49)$$

Equation (2.49) is polynomial equation and it can be satisfied only when coefficient of each power of x is equal to zero. Equating higher powers of x to zero , we get

$$a_0[k(k+1) - n(n+1)] = 0 \qquad or \qquad a_0(k-n)(k+n+1) = 0$$

Since $a_0 \neq 0$, therefore $k = n \ or \ k = -(n+1)$, then equating the coefficient of x^{k-1} to zero, we get

$$a_1[k(k-1) - n(n+1)] = 0 \qquad or \qquad a_1(k+n)(k-n-1) = 0$$

For both cases $k = n \ and \ k = -(n+1)$, we have $(k+n)(k-n-1) \neq 0$, $\therefore \ a_1 = 0$

Then equating the coefficient of x^{k-r} to zero, we get

$$a_{r-2}(k-r+2)(k-r+1) - a_r[(k-r)(k-r+1) - n(n+1)] = 0$$

So that

$$a_r = -\frac{(k-r+1)(k-r+2)}{n(n+1) - (k-r)(k-r+1)} a_{r-2} \qquad (2.50)$$

$a_1 = 0$, equation (6) gives $a_3 = a_5 = a_7 \cdots = 0$,

*Case 1.*When $k = n$, we have

$$a_r = -\frac{(n-r+1)(n-r+2)}{n(n+1) - (n-r)(n-r+1)} a_{r-2}$$

$$= -\frac{(n-r+1)(n-r+2)}{r(2n-r+1)} a_{r-2}$$

So that

$$a_2 = -\frac{n(n-1)}{2(2n-1)} a_0$$

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)} a_2 = \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} a_0$$

$$a_6 = -\frac{(n-4)(n-5)}{6(2n-5)} a_4 = -\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2 \cdot 4 \cdot 6(2n-1)(2n-3)(2n-5)} a_0$$

………………………

$$a_{2r} = (-1)^r \frac{n(n-1)(n-2)(n-3)\cdots\cdots(n-2r+1)}{2\cdot 4\cdot 2r(2n-1)(2n-3)\cdots\cdots(2n-2r+1)} a_0 \qquad (2.51)$$

Consider

$$a_0 = \frac{(2n)!}{2^n n!\, n!} \qquad (2.52)$$

Put (2.52) in (2.51) we get

$$a_{2r} = (-1)^r \frac{n(n-1)(n-2)(n-3)\cdots\cdots(n-2r+1)(2n)!}{2\cdot 4\cdot 2r(2n-1)(2n-3)\cdots\cdots(2n-2r+1)2^n n!\, n!}$$

$$= (-1)^r \frac{n(n-1)(n-2)(n-3)\cdots\cdots(n-2r+1)(n-2r)!\,(2n)!}{2^{r+n} r!\,(2n-1)(2n-3)\cdots\cdots(2n-2r+1)(n-2r)!\, n!\, n!}$$

$$= (-1)^r \frac{n!\,(2n)!\,2n(2n-2)\cdots\cdots(2n-2r+2)}{2^{r+n} r!\,2n(2n-1)\cdots\cdots(2n-2r+2)(2n-2r+1)(n-2r)!\, n!\, n!}$$

$$= (-1)^r \frac{(2n)!\,2^r n(n-1)\cdots\cdots(n-r+1)(2n-2r)!}{2^{r+n} r!\, n!\,(2n)!\,(n-2r)!}$$

$$= (-1)^r \frac{n(n-1)\cdots\cdots(n-r+1)(n-r)!\,(2n-2r)!}{2^n r!\, n!\,(n-2r)!\,(n-r)!}$$

$$= (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)!} \qquad (2.53)$$

Since $a_1 = a_3 = a_5 = a_7 = \cdots\cdots\cdots = 0$ then equation (2.46) can be written as

$$y = \sum_{r=0}^{\infty} a_{2r}\, x^{n-2r} a_0 \neq 0 \qquad (2.54)$$

Substituting (2.53) in (2.54) we get solution of Legendre differential equation, denoted by $P_n(x)$, ie.,

$$y = P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)!} x^{n-2r} \qquad (2.55)$$

$P_n(x)$is known as Legendre polynomial of first kind. The factorial function cannot be a negative number, therefore the upper limit for r is changed from infinity to N

Case 2: when $k = -(n+1)$, equation (6) becomes

$$a_r = \frac{(n+r)(n+r-1)}{r(2n+r+1)} a_{r-2}$$

So that

$$a_2 = \frac{(n+2)(n+1)}{2(2n+3)} a_0$$

$$a_4 = \frac{(n+4)(n+3)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4(2n+3)(2n+5)} a_0$$

$$a_6 = \frac{(n+6)(n+5)}{6(2n+7)} a_4 = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{2 \cdot 4 \cdot 6(2n+3)(2n+5)(2n+7)} a_0$$

…………………

$$a_{2r} = \frac{(n+1)(n+2)\cdots\cdots(n+2r)}{2 \cdot 4 \cdot 6 \cdots 2r(2n+3)(2n+5)\cdots(2n+2r+1)} a_0 \qquad (2.56)$$

Consider

$$a_0 = \frac{2^n n!\, n!}{(2n+1)!} (2.57)$$

Put (2.57) in (2.56) we get

$$a_{2r} = \frac{(n+1)(n+2)\cdots\cdots(n+2r)2^n n!\, n!}{2 \cdot 4 \cdot 6 \cdots 2r(2n+3)(2n+5)\cdots(2n+2r+1)(2n+1)!}$$

$$= \frac{(n+2r)!\, 2^n n!\, (2n+2)(2n+4)\cdots\cdots(2n+2r)}{2^r r!\, (2n+2)(2n+3)\cdots(2n+2r)(2n+2r+1)(2n+1)!}$$

$$= \frac{(n+2r)!\, 2^n n!\, (n+1)(n+2)\cdots\cdots(n+r)2^r}{2^r r!\, (2n+2r+1)!} = \frac{2^n (n+2r)!\, (n+r)!}{r!\, (2n+2r+1)!} \qquad (2.58)$$

Since $a_1 = a_3 = a_5 = a_7 = \cdots\cdots\cdots = 0$ then equation (2) can be written as

$$y = \sum_{r=0}^{\infty} a_{2r}\, x^{-n-1-2r} a_0 \neq 0 \qquad (2.59)$$

Put (2.58) in (2.59) w get the solution of Legendre differential equation, denoted by $Q_n(x)$ is

$$y = Q_n(x) = \sum_{r=0}^{\infty} \frac{2^n (n+2r)!\,(n+r)!}{r!\,(2n+2r+1)!} x^{-n-1-2r}$$

$Q_n(x)$is known as Legendre polynomial of second kind.

The linear combination of $P_n(x)$ and $Q_n(x)$ is a solution of the Legendre differential equation.

$$y = AP_n(x) + BQ_n(x)$$

A and B are two arbitrary constants.

2.7.1 Some specific cases for$P_n(x)$:

(i) $P_0(x) = 1$

(ii) $P_1(x) = x$

(iii) $P_2(x) = \frac{1}{2}(3x^2 - 1)$

(iv) $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

(v) $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

(vi) $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

(vii) $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

(viii) $P_n(x) = (-1)^n P_n(-x) \quad or \quad P_n(-x) = (-1)^n P_n(x)$

(ix) $P_n(1) = 1$

(x) $P_n(-1) = (-1)^n$

(xi) $P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$

(xii) $P_{2n+1}(0) = 0$

(xiii) $P_n = P_m$

2.7.2 Generating function of$P_n(x)$:

The function $(1 - 2xt + t^2)^{-1/2}$ is known as generating function for the Legendre function $P_n(x)$. $P_n(x)$is the coefficient of t^nin the expansion of the function. ie.,

$$(1 - 2xt + t^2)^{-1/2} = \sum P_n(x)\, t^n \qquad (2.60)$$

Proof: $(1-2xt+t^2)^{-1/2} = [1-(2xt-t^2)]$

$$= 1 + \frac{1}{2}(2xt\ -t^2) + \frac{1\cdot 3}{2^3 2!}(2xt-t^2)^2\ +\ \frac{1\cdot 3\cdot 5}{2^3 3!}(2xt-t^2)^3 + \cdots$$

$$= 1 + \sum_{r=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2r-1)}{2^r\ r!}(2xt-t^2)^r \qquad (2.61)$$

Then

$$(2xt-t^2)^r\ =\ \sum_{k=0}^{r}\frac{r!}{k!\,(r-k)!}(2xt)^{r-k}(-t^2)^k$$

$$= \sum_{k=0}^{r}\frac{(-1)^k r!}{k!\,(r-k)!}(2x)^{r-k}t^{r+k} \qquad (2.62)$$

Put (2.62) in (2.61), we get

$$(1-2xt+t^2)^{-1/2} = 1 + \sum_{r=1}^{\infty}\sum_{k=0}^{r}\frac{(-1)^k 1\cdot 3\cdot 5\cdots(2r-1)}{2^r k!\,(r-k)!}(2x)^{r-k}t^{r+k}$$

$$= \sum_{r=0}^{\infty}\sum_{k=0}^{r}\frac{(-1)^k(2r)!}{2^{2r}\ r!\,k!\,(r-k)!}(2x)^{r-k}t^{r+k}$$

It shows that $0 \le k \le r \quad and \quad r \ge 0$ put $r\ \to n-k$ then the conditions are

$$0 \le k \le (n-k) \quad and \quad (n-k) \ge 0$$

The condition $0 \le k \le (n-k)$ is equivalent to $0 \le k \le n/2$

For this condition $0 \le k \le n/2$, when $k = 0, \ \ n \ge 0$

$and\ when\ \ k = n/2,\ \ (n-n/2) \ge 0\ \ or\ \ n \ge 0$

Let $N = n/2 \quad or \quad (n-1)/2\ \ whichever\ is\ an\ integer$

$$(1-2xt+t^2)^{-1/2}\ =\ \sum_{n=0}^{\infty}\sum_{k=0}^{N}\frac{(-1)^k(2n-2k)!\,(2x)^{n-2k}}{2^{2n-2k}(n-k)!\ \ k!\,(n-2k)!}t^n$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{N}\frac{(-1)^k(2n-2k)!\,x^{n-2k}}{2^n(n-k)!\,k!\,(n-2k)!}\right]t^n\ =\ \sum_{n=0}^{\infty}P_n(x)t^n \qquad (2.63)$$

ie., the coefficient of t^n in the expansion is $P_n(x)$

2.7.3 Recurrence relations for $P_n(x)$

Relations among various orders of Legendre function are known as recurrence relations.

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{2.64}$$

Differentiating with respect to t we get

$$-\frac{1}{2}(-2x+2t)(1-2xt+t^2)^{-3/2} = \sum P_n(x)nt^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum P_n(x)nt^{n-1} \tag{2.65}$$

Equation (2.64) in (2.65) ,we get

$$(x-t)\sum P_n(x)t^n = (1-2xt+t^2)\sum P_n(x)nt^{n-1} \tag{2.66}$$

Equating the coefficients of t^m on both sides of (2.66) we get

$$xP_m(x) - P_{m-1}(x) = (m+1)P_{m+1}(x) - 2xmP_m(x) + (m-1)P_{m-1}(x)$$

Re arranging the equation we get

$$(2m+1)xP_m(x) = (m+1)P_{m+1}(x) + mP_{m-1}(x) \tag{2.67}$$

(ii) when m is replaced by $m-1$ in equation (2.67) we get

$$(2m-1)xP_{m-1}(x) = mP_m(x) + (m-1)P_{m-2}(x)$$

Then we get

$$mP_m(x) = (2m-1)xP_{m-1}(x) - (m-1)P_{m-2}(x)$$

(iii) Differentiating equation (2.64) with respect to x, we get

$$-\frac{1}{2}(-2t)(1-2xt+t^2)^{-3/2} = \sum P'_n(x)t^n$$

$$t(1-2xt+t^2)^{-3/2} = \sum P'_n(x)t^n$$

$$t(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum P'_n(x)t^n \tag{2.68}$$

$put\ (5) in\ (9) we\ get,\qquad t\sum P_n(x)\,t^n = (1-2xt+t^2)\sum P'_n\,(x)t^n \qquad (2.69)$

Equating the coefficients of t^m in (2.69), we get

$$P_{m-1}(x) = P'_m(x) - 2xP'_{m-1}(x) + P'_{m-2}(x) \qquad (2.70)$$

(iv) Replacing $m \quad by \quad m+1$ in (2.70) we get

$$P_m(x) = P'_{m+1}(x) - 2xP'_m(x) + P'_{m-1}(x)$$

Rearrange the equation as

$$P'_{m+1}(x) + P'_{m-1}(x) = P_m(x) + 2xP'_m(x) \qquad (2.71)$$

(v) Differentiating equation (2.67) with respect to x, we get

$$(2m+1)xP'_m + (2m+1)P'_m = (m+1)P'_{m+1}(x) + mP'_{m-1}(x) \qquad (2.72)$$

Multiplying equation (2.72) $by\ 2$, $equation$ (2.71) $by\ (2m+1)$ and then adding ,we get

$$(2m+1)P_m(x) + P'_{m-1}(x) = P'_{m+1}(x) \qquad (2.73)$$

(vi) Replacing $m \quad by \quad m-1\ in$ (2.73), we get

$$(2m-1)P_{m-1}(x) + P'_{m-2}(x) = P'_m(x) \qquad (2.74)$$

2.8 Hermite Function

Hermite differential equation is

$$\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2ny = 0 \qquad (2.75)$$

Where n is positive integer. Solution for this equation is known as Hermite function. To find singular points and series solution of the equation, consider $P(x) = -2x \quad and \quad Q(x) = 2n$ There are no singular points. By Fuchs theorem, Hermite differential equation has a series solution

$$y = \sum_{m=0}^{\infty} a_m x^{k-m} a_0 \neq 0 \qquad (2.76)$$

On differentiating equation (2.76) with respect to x, we get

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} a_m\,(k-m)x^{k-m-1} \qquad (2.77)$$

And

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} a_m\,(k-m)(k-m-1)x^{k-m-2} \qquad (2.78)$$

Using (2.76), (2.77) and (2.78) in (2.75), we get

$$\sum a_m\,(k-m)(k-m-1)x^{k-m-2} + \sum a_m\,2(n-k+m)x^{k-m} = 0 \qquad (2.79)$$

(2.79) is polynomial equation. Equating the coefficient of the highest power of x to zero, ie., $a_0(n-k) = 0$since $a_0 \neq 0, k = n$, then equating the coefficient of x^{k-1} to zero,ie., $a_1(n-k+1) = 0$ For $k = n$ we have $(n-k+1) \neq 0$ and therefore, $a_1 = 0$. Further equating the coefficient of x^{k-r} to zero, we get

$$a_{r-2}(k-r+2)(k-r+1) + 2a_r(n-k+r) = 0$$

$$a_r = -\frac{(k-r+2)(k-r+1)}{2(n-k+r)}a_{r-2} \qquad (2.80)$$

Since $a_1 = 0$ equation (6) gives $a_3 = a_5 = a_7 \cdots\cdots = 0$ then for $k = n$, we have

$$a_r = -\frac{(n-r+2)(n-r+1)}{2r}a_{r-2}$$

$$a_2 = -\frac{n(n-1)}{2\cdot 2}a_0$$

$$a_4 = -\frac{(n-2)(n-3)}{2\cdot 4}a_2 = \frac{n(n-1)(n-2)(n-3)}{2^4 2!}a_0$$

$$a_6 = -\frac{(n-4)(n-5)}{2\cdot 6}a_4 = -\frac{n(n-1)\cdots(n-5)}{2^6 3!}a_0$$

$\cdots\cdots\cdots\cdots\cdots$

$$a_{2r} = (-1)^r\frac{n(n-1)\cdots(n-2r+1)}{2^{2r}r!}a_0 = \frac{(-1)^r n!}{2^{2r}r!\,(n-2r)!}a_0 \qquad (2.81)$$

Let $a_0 = 2^n$ and substitute in (2.81) we get

$$a_{2r} = \frac{(-1)^r n!}{2^{2r-n}r!\,(n-2r)!} \qquad (2.82)$$

Since $a_1 = a_3 = a_5 = a_7 \cdots\cdots = 0$ then equation (2.76) can be written as

$$y = \sum_{r=0}^{\infty} a_{2r}x^{n-2r} a_0 \neq 0 \qquad (2.83)$$

Substitute (8) in (9) we get solution of Hermite differential equation denoted by $H_n(x)$

$$y = H_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r n!}{2^{2r-n} r!\,(n-2r)!} x^{n-2r} = \sum_{r=0}^{N} \frac{(-1)^r n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

2.8.1 Some Specific cases for $H_n(x)$

(i) $H_n(x) = 1$

(ii) $H_1(x) = 2x$

(iii) $H_2(x) = 4x^2 - 2$

(iv) $H_3(x) = 8x^3 - 12x$

(v) $H_4(x) = 16x^4 - 48x^2 + 12$

(vi) $H_5(x) = 32x^5 - 160x^3 + 120x$

(vii) $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

(viii) $H_n(0) = \frac{(-1)^{n/2} n!}{(n/2)!}$ when n is even integer

(ix) $H_n(0) = 0$ when n is odd integer

(x) $H_n(-x) = (-1)^n H_n(x)$

2.8.2 Generating function for $H_n(x)$

The function e^{2xt-t^2} is known as generating function for Hermite function. The coefficient of t^n in the expansion is $\frac{H_n(x)}{n!}$ ie., $e^{2xt-t^2} = \sum \frac{H_n(x)}{n!} t^n$

Proof: we know that

$$e^{2xt} = \sum_{r=0}^{\infty} \frac{(2x)^r t^r}{r!} \qquad (2.84)$$

And

$$e^{-t^2} = \sum_{s=0}^{\infty} \frac{(-1)^s t^{2s}}{s!} \qquad (2.85)$$

The product of (2.84) and (2.85) is

$$e^{2xt}e^{-t^2} = \sum_{r=0}^{\infty}\frac{(2x)^r t^r}{r!}\sum_{s=0}^{\infty}\frac{(-1)^s t^{2s}}{s!}$$

$$e^{2xt-t^2} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s (2x)^r t^{r+2s}}{s!\,r!} \qquad (2.86)$$

Put $r = n - 2s$ in (2.86) we get

$$e^{2xt-t^2} = \sum_{n=2s}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s (2x)^{n-2s} t^n}{s!\,(n-2s)!}$$

$$= \sum_{n}\frac{1}{n!}\left[\sum_{s=0}^{n/2}\frac{(-1)^s n!\,(2x)^{n-2s} t^n}{s!\,(n-2s)!}\right]t^n = \sum_{n}\frac{H_n(x)}{n!}t^n \qquad (2.87)$$

This shows that the coefficient of t^n of the expansion of e^{2xt-t^2} is $\frac{H_n(x)}{n!}$

2.8.3 Recurrence relations for $H_n(x)$

The relations among various orders of the Hermite function are known as recurrence relations

(i) We know that

$$e^{2xt-t^2} = \sum_{n}\frac{H_n(x)}{n!}t^n \qquad (2.88)$$

Differentiating equation (2.88) with respect to t, we get

$$e^{2xt-t^2}(2x-2t) = \sum_{n}\frac{H_n(x)}{n!}\,n\,t^{n-1} \qquad (2.89)$$

Substitute (2.88) in (2.89) we get

$$2(x-t)\sum_{n}\frac{H_n(x)}{n!}t^n = \sum_{n}\frac{H_n(x)}{n!}\,n\,t^{n-1} \qquad (2.90)$$

Equating the coefficient of t^m on both sides of (2.90) we get

$$2x\frac{H_m(x)}{m!} - 2\frac{H_{m-1}(x)}{(m-1)!} = \frac{H_{m+1}(x)}{m!}$$

$$2xH_m(x) = H_{m+1}(x) + 2mH_{m-1}(x) \qquad (2.91)$$

(ii) Differentiating equation (2.88) with respect to x, we get

$$e^{2xt-t^2}2t = \sum \frac{H'_n(x)}{n!} t^n \qquad (2.92)$$

Substitute equation (2.88) in (2.92) we get

$$2t\sum_n \frac{H_n(x)}{n!} t^n = \sum \frac{H'_n(x)}{n!} t^n \qquad (2.93)$$

Equating the coefficient of t^m on both of (2.93) we get

$$2\frac{H_{m-1}(x)}{(m-1)!} = \frac{H_m^{'}(x)}{m!} \qquad or \qquad 2mH_{m-1}(x) = H_m^{'}(x) \qquad (2.94)$$

(iii) Equating the coefficient t^0 we get $H_m^{'}(x) = 0$

(iv) Substitute the value of $2mH_{m-1}(x)$ from equation (2.94) in (2.91) we get

$$2xH_m(x) = H_{m+1}(x) + H_m^{'}(x)$$

UNIT IV : FOURIER AND LAPLACE TRANSFORM

Fourier transform-properties of Fourier transform-convolution – Fourier cosine and sine transform-Fourier transform of derivatives- Application of Fourier transform-vibrations in a string-Laplace transform-inverse Laplace transform- Application of Laplace transform-Simple Harmonic motion

3.1 Fourier Transform:

The Fourier transform $g(k)$of a function$f(x)$ is defined by the equation

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

It is denoted by $g(k) \; = F\{f(x)\}$

The equation which gives (x) , for a known value of $g(k)$ is called the inverse of Fourier Transform ie., $f(x) = \mathrm{F}^{-1}\{\mathrm{g(k)}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dx.$

Example1: Find the Fourier transform of Gaussian function $f(x) = e^{-x^2}$

Solution:

By definition of Fourier transform we have, $g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$

$$\therefore\, g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, e^{-ikx}\, dx$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2-ikx}\, dx$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ikx)}\, dx$$

$$x^2 + ikx \;=\; \left(x + \frac{ik}{2}\right)^2 - \left(\frac{ik}{2}\right)^2 = x^2 + 2x\frac{ik}{2} + \frac{i^2k^2}{4} - \frac{i^2k^2}{4} = x^2 + ikx$$

Then $g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{\left(x+\frac{ik}{2}\right)^2 - \left(\frac{ik}{2}\right)^2\right\}}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x+\frac{ik}{2}\right)^2} e^{\left(\frac{ik}{2}\right)^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4}} \int_{-\infty}^{\infty} e^{-\left(x+\frac{ik}{2}\right)^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4}} \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}}$$

Thus Fourier transform of a Gaussian function is another Gaussian function.

3.2 Properties of Fourier Transform:

3.2.1.Addition theorem or Linearity theorem:

If $f(t) = a_1 f_1(t) + a_2 f_2(t) + \cdots\cdots + a_n f_n(t)$ then the Fourier transform of $f(t)$ is given by $g(\omega) = a_1 g_1(\omega) + a_2 g_2(t) + \cdots\cdots + a_n g_n(\omega)$ where $g_1(\omega), g_2(\omega), \cdots\cdots g_n(\omega)$ are Fourier transform of $f_1(t), f_2(t), \cdots\cdots f_n(t)$ and $a_1, a_2, \cdots\cdots, a_n$ are constants.

Proof:

The Fourier transform of $f(t)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$\therefore g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t) + \cdots\cdots + a_n f_n(t)]\, e^{-i\omega t}\, dt$$

$$g(\omega) = a_1 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t)\, e^{-i\omega t}\, dt + a_2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t)\, e^{-i\omega t}\, dt + \cdots\cdots$$

$$+ a_n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(t)\, e^{-i\omega t}\, dt$$

ie.,$g(\omega) = a_1 g_1(\omega) + a_2 g_2(\omega) + \cdots\cdots + a_n g_n(\omega)$ Hence proved.

3.2.2. Similarity theorem or change of scale property:

If $g(\omega)$ is the Fourier transform of $f(t)$, then Fourier transform of $f(at)$ is $\frac{1}{a}\, g(\omega)$.

Proof:

$$We\ have, \qquad F.T\,\{f(t)\} = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$Then\ F.T\ \{f(at)\} = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt$$

$$Put \quad y = at\,, \quad \therefore \quad t = \frac{y}{a} \quad and \quad dt = \frac{dy}{a}$$

$$Therefore\ F.T\ \{f(at)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at)\, e^{-i\omega \frac{y}{a}} \frac{dy}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\left(\frac{w}{a}\right)y}\, dy \;\; = \frac{1}{a} g\left(\frac{\omega}{a}\right)$$

<u>3.2.3. Fourier transform of the complex conjugate:</u>

If $g(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of the complex conjugate of $f(t)$ will be given by $g^*(-\omega)$; where * indicates the complex conjugate of the corresponding complex function.

Proof:

$$We\ have, g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$Taking\ complex\ conjugate\ on\ both\ sides\ g^*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(t)\, e^{i\omega t}\, dt$$

$$Replacing\ \ \omega by - \omega, we\ get\ g^*(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(t)\, e^{-i\omega t}\, dt$$

Therefore $g^*(-\omega) = F.T.[f^*(t)]$

<u>3.2.4. Shifting Property:</u>

If $g(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $f(t \pm a)$ will be given by $e^{\pm i\omega a}\, g(\omega)$ where a is any constant.

Proof:

$$F.T.[f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t \pm a)\, e^{-i\omega t}\, dt$$

Put $(t \pm a) = y$; $t = y \mp a$; and $dt = dy$

$$F.T.[f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\omega(y \mp a)} dy$$

$$= e^{\mp i\omega a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\omega y}\, dy$$

$$= e^{\mp i\omega a}\, g(\omega)$$

ie., If a function be shifted in the positive or negative direction by an amount a, no Fourier component changes in amplitude, but its Fourier transform suffers phase changes.

3.2.5. Modulation Theorem:

If $g(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $f(t)\cos at$ is given by $\frac{1}{2} g(\omega - a) + \frac{1}{2}\, g(\omega + a)$.

$$F.T.[f(t)cosat] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)cosat\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)(\frac{e^{iat} + e^{-iat}}{2})e^{-i\omega t}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{\infty} f(t)e^{-i(\omega-a)}\, dt + f(t)e^{-i(\omega+a)} dt\,\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i(\omega-a)}\, dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega+a)} dt\right]$$

$$= \frac{1}{2}[g(\omega - a) + g(\omega + a)]$$

Convolution: The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x)g(x-a)da$$

3.2.6. Convolution Theorem:

The Fourier transform of the convolution of $f(x)$ and $k(x)$ is the product of their Fourier transforms. ie., $F\{f(x) * k(x)\} = F\{f(x)\} \cdot F\{k(x)\}$

The convolution Theorem involving Fourier Transform: An integral, $I(x)$ of the form

$$I(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)k(\xi)\, d\xi$$

$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)k(x-\xi)\, d\xi$ *is known as a convolution integral in the interval* (-∞, +∞)

Taking the Fourier transform of $I(x), F\{I(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(x) e^{-ikx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)k(x-\xi) d\xi$$

As $e^{ikx} e^{-ikx} = e^0 = 1,$ *multiply the RHS of the above equation by* $e^{ikx} e^{-ikx}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x-\xi) e^{-ik(x-\xi)}\, dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-ikx}\, d\xi$$

In the 1st integral change to $x' = x - \xi$ *, we get*

$F\{I(x)\} = F\{k(x)\} \cdot F\{f(x)\}$ This is the convolution theorem.

3.2.7. Pasevals Theorem:

The Fourier transform of a convolution integral is given by the product of transform of the convolving functions. Let $f(t)$ be given convolution integral

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t') f_2(t-t') e^{-i\omega t}\, dt$$

The Fourier transform of $f(t)$ *is*

$$g(\omega) = F.T.[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t')\, f_2(t-t')\ dt'dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(t')e^{-i\omega t}dt' \int_{-\infty}^{\infty} e^{-i\omega t}\ e^{i\omega t'} f_2(t-t')\ dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(t')e^{-i\omega t'}\ dt' \int_{-\infty}^{\infty} e^{-i\omega t}\ e^{i\omega t'} f_2(t-t')\ dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(t')e^{-i\omega t'}\ dt' \int_{-\infty}^{\infty} e^{-i\omega (t-t')} f_2(t-t')\ dt$$

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t')\, e^{-i\omega t'} dt'$$

$$g_2(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(t-t')\, e^{-i\omega (t-t')} dt'$$

$Put\ \ t\ =\ t - t'$ $\quad g_2(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(t)\, e^{-i\omega t}\, dt$

$$g(\omega) = g_1(\omega)\, g_2(\omega)$$

3.2.8. Derivative of Fourier Transform:

$$If\, g(\omega) is\ the\ Fourier\ transform\ of\ f(t)\ then\ g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Differentiating on both sides with respect to ω, we get

$$\frac{dg(\omega)}{d\omega} = \frac{1}{\sqrt{2\pi}}\frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\quad dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial\omega}\left[f(t)e^{-i\omega t}\right]\ dt$$

$$= -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\, f(t)e^{-i\omega t}\quad dt$$

$$= -i\, F.T.[tf(t)]$$

$If\ we\ differentiate\ n\ times\ w.r.to\ \ \omega\ we\ get,\quad \frac{d^n g}{d\omega} = (-1)^n F.T.[t^n\ f(t)]$

3.2.9. Fourier transform of a Derivative:

Let $g_1(\omega)$ be the Fourier transform of the first derivative function $f(t)$, then

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dt} e^{-i\omega t}\, dt$$

$Integrating\ by\ parts\ \int u\, dv = uv - \int vdu$

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t} f(t)\right]_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}}[0] + i\omega\ \ g(\omega)$$

$g_1(\omega)\ =\ i\omega\, g(\omega)$

$ie.,\ \ F.T.of\ \frac{df}{dt}\quad = \ i\omega\, F.T.of\ f(t)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dt} e^{-i\omega t}\, dt$$

$$= i\omega\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dtReplacing\ f(t)\ \ by\ \frac{df}{dt}\ on\ both\ sides\ ,we\ get$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^2 f}{dt^2} e^{-i\omega t}\, dt = i\omega\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dt} e^{-i\omega t}\, dt$$

$$= (i\omega)^2 g(\omega)$$

$ie.,\quad F.T.of\ \frac{d^2 f}{dt^2}\quad =\ \ (i\omega)^2\ F.T.of\ f(t)$

$Repeating\ these\ process\ n\ times\ we\ get,\qquad F.T.of\ \frac{d^n f}{dt^n}\ \ =\ \ (i\omega)^n\ F.T.of\ f(t)$

3.2.10. Fourier sine and cosine Transform of Derivatives:

The Fourier sine and cosine Transform of a function $f(t)$ are defined as

$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) sin\omega t\, dt$ and $g_c(\omega)$

$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) cos\omega t\, dt$ *Fourier sine transform of* 1*st derivative* $\frac{df}{dt}$ *is* $g_{1s}(\omega)$

$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dt} sin\omega t\, dt$

$g_{1s}(\omega) = \sqrt{\frac{2}{\pi}} [f(t) sin\omega t]_0^{\infty} - \sqrt{\frac{2}{\pi}}\ \omega \int_0^{\infty} f(t) cos\omega t\, dt$

$g_{1s}(\omega) = 0 - \omega \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) cos\omega t\, dt$

$$g_{1s}(\omega) = -\omega\, g_c(\omega)$$

Fourier cosine transform of 1*st derivative* $\frac{df}{dt}$ *is* $g_{1c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dt} cos\omega t\, dt$

$g_{1c}(\omega) = \sqrt{\frac{2}{\pi}} [f(t) cos\omega t]_0^{\infty} + \sqrt{\frac{2}{\pi}}\ \omega \int_0^{\infty} f(t) sin\omega t\, dt$

$g_{1c}(\omega) = -\sqrt{\frac{2}{\pi}}\ f(0) + \omega \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) sin\omega t\, dt\, g_{1c}(\omega) = \omega\, g_s(\omega) - \sqrt{\frac{2}{\pi}}\ f(0)$

Then Fourier cosine transform of 1*st derivative*

$\frac{d^2 f}{dt^2}$ *is* $g_{2s}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{d^2 f}{dt^2} sin\omega t\, dt$

$g_{2s}(\omega) = \sqrt{\frac{2}{\pi}} \left[\frac{df}{dt}\ sin\omega t\right]_0^{\infty} - \sqrt{\frac{2}{\pi}}\ \omega \int_0^{\infty} \frac{df}{dt}\ cos\omega t g_{2s}(\omega) = 0 - \omega \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dt} cos\omega t\, dt$

$$g_{2s}(\omega) = -\omega\, g_{1c}(\omega)$$

$$g_{2s}(\omega) = -\omega \left[\omega\, g_s(\omega) - \sqrt{\frac{2}{\pi}}\ f(0)\right]$$

$g_{2s}(\omega) = -\omega^2 g_s(\omega) + \sqrt{\frac{2}{\pi}}\omega f(0)$

Similarly , we can find

$g_{2c}(\omega) = -\omega^2 g_c(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$

Problems:

1. Find the Fourier transform of $e^{-|t|}$

Solution:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Given that $f(t) = e^{-|t|}$ $\quad \therefore g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|}\, e^{-i\omega t}\, dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t}\, e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t}\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t(1-i\omega)}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t(1+i\omega)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{t(1-i\omega)}}{1-i\omega}\right]_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-t(1-i\omega)}}{1+i\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\omega^2}\right) = \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2}\right)$$

2. Write the Fourier transform of the function $f(t) and\ hence\ prove\ moment\ theorem,$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{m_n}{n!} (-i\omega)^2$$

$$where\ \ m_n = \int_{-\infty}^{\infty} t^n f(t) dt\ \ and\ is\ known\ as\ moment\ of\ f(t).$$

Solution:

$$The\ F.T. of\ f(t)\ \ is,\ \ g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[1 - \frac{i\omega t}{1!} + \frac{(i\omega t)^2}{2!} - \frac{(i\omega t)^3}{3!} + \cdots \right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sum_{n=0}^{n} \frac{(-i\omega t)^n}{n!} dt$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{n} \frac{(-i\omega)^n}{n!} \int_{-\infty}^{\infty} t^n f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{n} \frac{(-i\omega)^n}{n!} m_n \qquad\qquad Hence\ proved\ the\ moment\ theorem.$$

3. Find the Fourier transform of the slit function $f(x)$ defined as $f(x) = \begin{cases} \frac{1}{\epsilon}, & |x| \le \epsilon \\ 0, & |x| > \epsilon \end{cases}$

Determine the limit of this transform as $\epsilon \to 0$ and discuss the result.

Solution:

$$The\ F.T. of\ f(x)\ \ is,\ \ g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\epsilon} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-\epsilon}^{\epsilon}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \left[\frac{e^{i\omega\epsilon} - e^{-i\omega\epsilon}}{i\omega} \right]$$

Multiply and divide by 2 we get,

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{\epsilon\omega}\left[\frac{e^{i\omega\epsilon} - e^{-i\omega\epsilon}}{2i}\right]$$

$$= \frac{\sin\omega\epsilon}{\omega\epsilon}$$

$\lim_{\epsilon\to 0} g(\omega) = \lim_{\epsilon\to 0}\sqrt{\frac{2}{\pi}}\frac{\sin\omega\epsilon}{\omega\epsilon}$ it is in the $\frac{0}{0}$ form

$$\therefore \quad = \lim_{\epsilon\to 0}\sqrt{\frac{2}{\pi}}\frac{\frac{\partial}{\partial\epsilon}(\sin\omega\epsilon)}{\frac{\partial}{\partial\epsilon}(\omega\epsilon)}$$

$$= \lim_{\epsilon\to 0}\sqrt{\frac{2}{\pi}}\frac{\omega\ cos\omega\epsilon}{\omega} = \sqrt{\frac{2}{\pi}}$$

$g(\omega)$approaches $\sqrt{\frac{2}{\pi}}$ as $\epsilon \to 0$, while the function itself approaches ∞ as $x \to 0$, then the function and its Fourier Transform are plotted.

4. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$

Solution:

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sin\omega x\ dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\ sin\omega x\ dx$$

Differentiating w. r. to ω, we get

$$\frac{dg_s(\omega)}{d\omega} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\ x\ cos\omega x\ dx$$

$$\frac{dg_s(\omega)}{d\omega} = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$$

Then integrating we get $g_s(\omega) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{\omega}{a}\right) + A$

$\omega = 0, \quad g_s(\omega) = g_s(0) = A \quad and\ g_s(\omega) = 0$ for $\omega = 0 \therefore \ A = 0$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\omega}{a}\right)$$

5. Find the cosine transform of a function of x which is unity for $0 < x < a$ and zero for $x \geq a$.

Solution:

Given that $f(x) = \begin{cases} 1 & 0 < x < 0 \\ 0, & x \geq a \end{cases}$

$$F_c\ .T.of\ f(x) is\ \ g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) cos\omega x\ dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a f(x)\ cos\omega x\ dx + \int_a^{\infty} f(x)\ cos\omega x\ dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a 1\ cos\omega x\ dx + \int_a^{\infty} 0\ cos\omega x\ dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{sin\omega a}{\omega}$$

<u>3.3. Application of Fourier Transform</u>: (Vibration in a string)

Consider an infinitely long freely vibrating string, let y be the displacement of vibration from its mean position and satisfies the wave equation

$$\frac{d^2y}{dx^2} = \frac{1}{v^2} \frac{d^2y}{dt^2} \qquad (3.1)$$

Where x is the distance measured along the String;

$v$is the velocity of wave moving along the string: and y is a function x and t

The initial condition of the string is $y(x, 0) = F(x)$

Multiplying on both sides of equation (1) by $\frac{e^{isx}}{\sqrt{2\pi}}$ and integrating over the limit $(-\infty, \infty)$ we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{isx}\,dx = \frac{1}{v^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}e^{isx}\,dx \qquad (3.2)$$

It is the Fourier Transform of second derivative

Let

$$Y(s,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\, e^{isx}\,dx \qquad (3.3)$$

$$\therefore \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{isx}\,dx = (-is)^2 Y(s,t) \qquad (3.4)$$

Equation (3.2) becomes $(-is)^2 Y(s,t) = \frac{1}{v^2}\frac{\partial^2 Y(s,t)}{\partial t^2}$ (3.5)

ie., $\frac{\partial^2 y}{\partial t^2} = -\, v^2 s^2 Y$ (3.6)

at t=0, equation (3.3) becomes $Y(s,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,o)\, e^{isx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\, e^{isx}\,dx = f(s) \qquad (3.7)$$

A general solution of equation (3.6) is $Y(s,t) = f(s)e^{\pm ivst}$ (3.8)

The inverse Fourier Transform of (3.3) is ,

$y(x,t)$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(s,t)\, e^{-isx}\,ds$$

(3.9) $Using\ (8)\ in\ (9), We\ get\quad y(x,t)$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{\pm ivst}\, e^{-isx}\,ds$$

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-is(x \mp vt)}\, ds \qquad (3.10)$$

$At\ t = 0, we\ have\ \ y(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-isx}\, ds \ = F(x)$

$F(x)$is the inverse Fourier transform of $f(s)$, therefore $y(x,t) = F(x \mp vt)$. This corresponds to the waves moving in + x and – x directions respectively.

3.4 Laplace Transform

If $F(t)$ be a function of (t) defined for all values of (t) , then Laplace transform of $F(t)$ is denoted by $\mathcal{L}\{F(t)\}$ or $F(s)$ or $f(s)$ is defined as

$$\mathcal{L}\{F(t)\} = \ F(s) = \ f(s) = \int_0^{\infty} F(t) e^{-st} \mathrm{dt}$$

The parameter (s) is real positive number and the integral exists.

If the integral converges for some value of (s), then only the Laplace transformation of $F(t)$ exists otherwise not. $\mathcal{L}$ is Laplace transformation operator. The operation of multiplying $F(t)$ by e^{-st} and then integrating between the limits 0 to ∞ is known as Laplace transformation.

3.4.1First Shifting Theorem:

$If\ \ \mathcal{L}\{F(t)\} \ = f(s)\,, then\ \ \mathcal{L}\{e^{at} F(t)\} \ = f(s-a)$

ie., if$f(s)$ is the Laplace transformation of the function $F(t)$ and a is any real or complex number then $f(s-a)$ is Laplace transformation of $e^{at} F(t)$.

$$f(s) = \ \mathcal{L}\{F(t)\} \ \Rightarrow f(s-a) \ = \ \mathcal{L}\{e^{at} F(t)\}$$

.

Proof:

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} F(t) e^{-st}\, \mathrm{dt}. \mathcal{L}\{e^{at} F(t)\} = \int_0^{\infty} e^{at} e^{-st} F(t) \mathrm{dt}$$

$$= \int_0^\infty e^{-(s-a)t} F(t)dt$$

$Put\ (s-a)\ u\ > 0,$ $\quad = \int_0^\infty e^{-ut} F(t)\, dt.$

$$= f(u)$$

$Replace\, u\ by\ f(s-a\,),\quad then\ L\{e^{at}F(t)\} = f(s-a)\ hence\, proved.$

3.4.2 Second Shifting Theorem:

If $\mathcal{L}\{F(t)\}$ $= f(s)$ and $G(t) = \begin{cases} F(t-a), & t > 0 \\ 0, & t < a \end{cases}$ then $L\{G(t)\}$ $= e^{-as} f(s)$

Proof :

$By\, definition,\quad \mathcal{L}\{G(t)\} \quad = \int_0^\infty G(t) e^{-st}\, dt.$

$$= \int_0^a G(t)e^{-st}\, dt \ + \int_a^\infty G(t)e^{-st}\, dt, \qquad 0 < a < \infty$$

$$= \int_0^a 0\ e^{-st}\, dt \ + \int_a^\infty F(t-a)e^{-st}\, dt$$

$$= \int_a^\infty F(t-a)e^{-st}\, dt$$

$Put\ (t-a) = u\,; t = u + a;\ dt = du\ when\ u = 0\ ,t = a\ and\, u = \infty\ ,t = \infty$

$$\therefore \mathcal{L}\{G(t)\} = \int_0^\infty F(u)e^{-s(u+a)}\, du = e^{-sa}\int_0^\infty F(u)e^{-su} du$$

$By\, properties\, of\ definite\ integrals\, we\, can\, write, \mathcal{L}\{G(t)\} \ = \ e^{-sa}\int_0^\infty F(t)e^{-st}\, du$

$= e^{-sa}L\{F(t)\}$ $=$ e^{-sa} $f(s)$ hence proved

3.4.3 Laplace Transform of derivatives:

If $\mathcal{L}\{F(t)\} = f(s)$ $then$ $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$; if $F(t)$ is continuous for $0 \le t \le N$ and of exponential order for $t > N$ while $F'(t)$ is sectionaly continuous for $0 \le t \le N$.

Proof:

Case1

$If\ F'(t)\ is\ continuous\ for\ all\ t \geq 0\ \ then\ \ = \int_0^\infty F'(t)e^{-st}\,dt$

$= [e^{-st}F(t)]_0^\infty - \int_0^\infty F(t)(-se^{-st})\,\mathrm{dt}$

$= \left[\lim_{t\to\infty}(e^{-st}F(t))\right] - F(0) + s\int_0^\infty F(t)e^{-st}\,\mathrm{dt}$

$= \left[\lim_{t\to\infty}(e^{-st}F(t))\right] - F(0) + sL\{F(t)\}$

$\lim_{t\to\infty}(e^{-st}F(t))$ =0 , for $s > a$

$\mathcal{L}\{F'(t)\} = sL\{F(t)\} - F(0)$

Case2 (i) If $F'(t)$ is merely piecewise continuous, then the integral can be broken into sum of integrals in different ranges from $0\ \ to\ \infty$ such that in each of such parts $F'(t)$ is continuous

We have $\mathcal{L}\{F'(t)\} = sL\{F(t)\} - F(0)$ and $\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0)$

Put $G(t) = F'(t)$, $\{F''(t)\} = s\mathcal{L}\{F'(t)\} - F'(0)$

$= s[s\mathcal{L}\{F(t)\} - F(0)]$ - $F'(0)$

$= s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0) = \mathcal{L}\{F''(t)\}$

(ii) $\mathcal{L}\{H''(t)\} = s^2\mathcal{L}\{H(t)\} - sH(0) - H'(0)$

Put $H(t) = F'(t)$, $\mathcal{L}\{F'''(t)\} = s^2\mathcal{L}\{F'(t)\} - sF'(0) - F''(0)$

$$= s^2[s\mathcal{L}\{F(t)\} - F(0)] - sF'(0) - F''(0)$$

$$\mathcal{L}\{F'''(t)\} = s^3\mathcal{L}\{F(t)\} - s^2F(0) - sF^{'}(0) - F''(0)$$

(iii) If $F'(t)$ and its first (n-1) derivatives are continuous, then proceeding as above we have the general case,

$$\mathcal{L}\{F^n(t)\} = s^n\mathcal{L}\{F(t)\} - s^{n-1}F(0) - s^{n-2}F^{'}(0) - \cdots\cdots\cdots\cdots - F^{n-1}(0)$$

3.4.4 Laplace Transform of Integral:

If $\mathcal{L}\{F(t)\} = f(s)$ $then\ \frac{1}{s}\, f(s) = \mathcal{L}\left\{\int_0^t F(u)\, du\right\}$

Proof: Let $G(t) = \int_0^t F(u)\, du$ then $G(0) = \int_0^t F(u)\, du = 0$

And $G'(t) = \frac{d}{dt}\left[\int_0^t F(u)\, du\right] = F(t)$

But we know that $\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0)$

$$\therefore \mathcal{L}\{F(t)\} = s\mathcal{L}\{G(t)\} - (0)$$

$$\frac{1}{s}\mathcal{L}\{F(t)\} = \mathcal{L}\{G(t)\}$$

$$\frac{1}{s}\mathcal{L}\{F(t)\} = \mathcal{L}\left\{\int_0^t F(u)\, du\right\}$$

Problems:

1. Find $\mathcal{L}\{F(1)\}$ if Laplace Transform of the function $F(t) = 1$

We have $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\, F(t)dt$

$$\therefore \mathcal{L}\{F(1)\} \quad = \int_0^\infty e^{-st}\, 1dt$$

$$= \left[\frac{e - st}{-s}\right]_0^\infty$$

$$= \frac{1}{-s}[e^{-\infty} - e^0]$$

$$= \frac{1}{-s}[0 - 1] = \frac{1}{s}$$

2. Find $\mathcal{L}\{t^n\}$ where n is positive integer if Laplace Transform of the function $F(t) = t^n$

We have $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\, F(t)dt$

$$\therefore \mathcal{L}\{t^n\} \quad = \int_0^\infty e^{-st}\, t^n\, dt$$

$$= t^n\left[\frac{e - st}{-s}\right]_0^\infty - \int_0^\infty n\, t^{n-1}\frac{e^{-st}}{s}dt$$

$$= \frac{1}{-s}\lim_{t\to\infty}\left[\frac{(t)^n}{e^{st}}\right] + \frac{n}{s}\int_0^\infty t^{n-1}\, e^{-st} dt$$

$$= 0 + \frac{n}{s}\int_0^\infty t^{n-1}\, e^{-st} dt$$

$$= \frac{n}{s}\left[\frac{(n-1)}{s}\int_0^\infty t^{n-2}\, e^{-st} dt\right]$$

$$= \frac{n}{s}\frac{(n-1)}{s}\left[\frac{(n-2)}{s}\int_0^\infty t^{n-3}\, e^{-st} dt\right]$$

Repeating for n times we get,

$$\frac{n!}{s^n}\int_0^\infty e^{-st}\, dt$$

$$= \frac{n!}{s^n}\left[\frac{e-st}{-s}\right]_0^\infty = \frac{n!}{s^n}\left[0 - \frac{1}{s}\right]_0^\infty = \frac{n!}{s^{n+1}}$$

3. Find$\mathcal{L}\{e^{at}\}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = e^{-\infty} - \frac{1}{-(s-a)} = 0 + \frac{1}{(s-a)} = \frac{1}{(s-a)}$$

4. Find $L\{\sin at\}$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i} \text{Then, } L\{\sin at\} = \int_0^\infty e^{-st}\frac{e^{iat} - e^{-iat}}{2i} dt$$

$$\frac{1}{2i}\left[\int_0^\infty e^{-st}\, e^{iat} dt - \int_0^\infty e^{-st}\, e^{-iat} dt\right]$$

$$= \frac{1}{2i}\left[\int_0^\infty e^{(ia-s)t}\, dt - \int_0^\infty e^{-(ia+s)t}\, dt\right]$$

$$= \frac{1}{2i}\left[\frac{e^{(ia-s)t}}{(ia-s)} + \frac{e^{-(ia-s)t}}{(ia+s)}\right]_0^\infty$$

$$= \frac{1}{2i}\left[0 - \left(\frac{1}{ia-s} + \frac{1}{ia+s}\right)\right]$$

$$= \frac{1}{2i}\left[-\left(\frac{ia+s+ia-s}{-a^2-s^2}\right)\right]$$

$$= \frac{1}{2i}\left[-\left(\frac{2ia}{-(a^2+s^{2)}}\right)\right]$$

$$= \frac{a}{(a^2+s^{2)}}$$

5. $Find\ \mathcal{L}\{e^{-t}(3\,sinh\,2t\ \ -5\,cosh\,2t)\}$

$\mathcal{L}\{F(t)\} = f(s) \quad then\ \mathcal{L}\{e^{at}F(t)\} = f(s-a)$

$\mathcal{L}\{\sinh 2t\} = \dfrac{2}{s^2-2^2}\,and\ \ \mathcal{L}\{cosh\,2t\} = \dfrac{s}{s^2-2^2}$

$3\mathcal{L}\{e^{-t}\sinh 2t\} - 5\mathcal{L}\{e^{-t}\cosh 2t\} = 3\ \dfrac{2}{(s+1)^2-2^2} - 5\ \dfrac{s+1}{(s+1)^2-2^2}$

$$= \frac{6-5(s+1)}{(s+1)^2-2^2}$$

$$= \frac{1-5s}{s^2+2s-3}$$

Problems: (for Second Shifting Property)

6. Find $\mathcal{L}\{F(t)\}$ if $F(t) \ = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right),\ t > \frac{2\pi}{3} \\ 0,\ x < \frac{2\pi}{3} \end{cases}$

$$\mathcal{L}\{F(t)\} = \int_0^{\frac{2\pi}{3}} e^{-st}\,F(t)dt + \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\,F(t)dt$$

$$= \int_0^{\frac{2\pi}{3}} e^{-st}\,0dt + \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$$

$Put\ t - \frac{2\pi}{3} = u,\ \ for\ t = \frac{2\pi}{3}\ , u = 0; and\ for\ t = \infty, u = \infty\ then\ \ t\ can\ be\ written\ as$

$$t = \ u - \frac{2\pi}{3}$$

$$\therefore \mathcal{L}\{F(t)\} = \int_0^\infty e^{-s\left(u+\frac{2\pi}{3}\right)} \cos u\, du$$

$$= e^{-\frac{2\pi s}{3}} \int_0^\infty e^{-su} \cos u\, du$$

$$= e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}$$

$$= \frac{s\, e^{-\frac{2\pi s}{3}}}{s^2 + 1}$$

7. Evaluate $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\, F(t) dt$$

$$= \int_0^1 e^{-st}\, 0\, dt + + \int_2^\infty e^{-st}\, 0\, dt$$

$$= \int_1^2 e^{-st}\, t\, dt$$

$$= \left[t\, \frac{e^{-st}}{-s}\right]_1^2 - \int_1^2 \frac{e^{-st}}{s} dt$$

$$= -\frac{1}{s}\left[2e^{-2s} - \ 1. e^{-s} - \left(\frac{e^{-st}}{-s}\right)_1^2\right]$$

$$= -\frac{1}{s}\left[2e^{-2s} - \ e^{-s} + \frac{1}{s}(e^{-2s} - \ e^{-s})\right]$$

$$= -\frac{2}{s} + \ \frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} e^{-s}$$

$$= \ \left(\frac{1}{s} + \frac{1}{s^2}\right) e^{-s} - e^{-2s}\left(\frac{2}{s} + \frac{1}{s^2}\right)$$

8. Derivative Problems: Evaluate (i) $\mathcal{L}\{1\} = \frac{1}{s}$ (ii) $\mathcal{L}\{t\} = \frac{1}{s^2}$ and (iii) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ using Laplace Transform of derivatives

(i) $\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} - F(0)$

Given that $F(t) = 1, \;\therefore F'(t) = 0, \; and \; F(0) = 1$

Substituting we get, $L\{0\} = sL\{1\} - 1$

$$0 = sL\{1\} - 1 \therefore L\{1\} = \frac{1}{s}$$

(ii) Given that $F(t) = t, \;\therefore F'(t) = 1, \; and \; F(0) = 0$

$\mathcal{L}\{1\} = s\mathcal{L}\{t\} - 0,$ but $\mathcal{L}\{1\} = \frac{1}{s}$

$\therefore \frac{1}{s} = s\mathcal{L}\{t\}$ and therefore $\mathcal{L}\{t\} = \frac{1}{s^2}$

(iii) $Given\ that \; F(t) = e^{at}, \;\therefore F'(t) = ae^{at}, \; and \; F(0) = 1$

$Substituting\ we\ get, \quad \mathcal{L}\{ae^{at}\} = s\mathcal{L}\{e^{at}\} - 1$

$a\,\mathcal{L}\{e^{at}\} = s\mathcal{L}\{e^{at}\} - 1$

$1 = s\mathcal{L}\{e^{at}\} - a\,\mathcal{L}\{e^{at}\}$

$ie., \quad \mathcal{L}\{e^{at}\}(s-a) = 1$

$\therefore \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

9. Using the derivative equation $\mathcal{L}\{F''(t)\} = s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0)$ show that

$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Solution:

Given that $F(t) = \sin at, \;\therefore F'(t) = a\cos at, \; F''(t) = -a\sin^2 at$;

$F(0) = 0, \quad F'(0) = a$

$\mathcal{L}\{-a^2\sin at\} = s^2\mathcal{L}\{\sin at\} - s(0) - a$

$-a^2\mathcal{L}\{\sin at\} = s^2\mathcal{L}\{\sin at\} - a$

$$\mathcal{L}\{\sin at\}(s^2 + a^2) = a$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + \text{a}^2}$$

10. Problem of Transform of integrals:

(i) Evaluate $\mathcal{L}\left\{\int_0^t \sin 2u\, du\right\}$

We have

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} = f(s)$$

$$\mathcal{L}\{F(u)du\} = \frac{f(s)}{s}$$

$$\therefore \mathcal{L}\left\{\int_0^t \sin 2u\, du\right\} = \frac{2}{s(s^2+4)}$$

$(ii)\quad Evaluate\quad \mathcal{L}\left\{\int_0^t \frac{sin\, t}{t}\, dt\right\}$

$We\ have\ \ \mathcal{L}\{F(t)\} = f(s), and\ \mathcal{L}\,\{F(u)du\} = \frac{f(s)}{s}\ \ and\ \ \mathcal{L}\left\{\frac{F(t)}{t}\, dt\right\} = \int\limits_s^\infty f(s)ds$

$$L\{\sin t\} = \frac{1}{s^2+1} = f(s)$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \int\limits_s^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1} s]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

<u>3.5 Inverse Laplace Transform</u>

<u>Partial fraction method</u>:

Any rational function $\frac{P(s)}{Q(s)}$where $P(s)$ and $Q(s)$ are polynomials with the degree of $P(s)$ less than that of $Q(s)$ can be written as the sum of rational functions (called partial fraction) having the form $\frac{A}{(as+b)^r}$, $\frac{As+B}{(as^2+bs+c)^r}$ where r=1,2,3,……… By finding the inverse Laplace transform of each of the partial fractions we can find $L^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$

Example:

1. $$\frac{2s-5}{(3s-4)(2s+1)^2} = \frac{A}{3s-4} + \frac{B}{(2s+1)^3} + \frac{C}{(2s+1)^2} + \frac{D}{2s+1}$$

2. $$\frac{3s^2-4s+2}{(s^2+2s+4)^2(s-5)} = \frac{As+B}{(s^2+2s+4)^2} + \frac{Cs+D}{s^2+2s+4} + \frac{E}{s-5}$$

<u>Inverse Laplace Transform definition:</u>

If the Laplace transform of a function F(t) is f(s) ie if $L\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace transform of $f(s)$.ie, $F(t) = L^{-1}\{f(s)\}$

Where L^{-1} is called the inverse Laplace transformation operator.

<u>Problems:</u>

1. $Find\ \ L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

$$\frac{3s+7}{s^2-2s+3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$3s+7 = A\,(s+1) + B\,(s-3)$

$$= (A+B)s + A - 3B$$

Equating the coefficient of s and constant terms we get

$A + B = 3; \qquad and \quad A - 3B = 7$

Solving these equations we get, $A = 4\ and\ B = -1$

$$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1} L^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = 4\,L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 4e^{3t} - e^{-t} \quad Because \quad L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}\ and\ L\{e^{at}\} = \frac{1}{s-a}$$

2. $Evaluate\ L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{s-3}$$

Another method to find the values A, B and C

Multiply both sides by $(s+1)$, and substitute $S \to -1$

$$\underset{s\to -1}{Lt}\frac{2s^2-4}{(s-2)(s-3)} = A + \frac{B(s+1)}{s-2} + \frac{C(s+1)}{s-3}$$

$$\frac{2(-1)^2-4}{(-1-2)(-1-3)} = A+0+0 \qquad \therefore \frac{-2}{12} = \frac{-1}{6} = A$$

Multiply both sides by $(s-2)$, and substitute $S \to 2$

$$\underset{s\to 2}{Lt}\frac{2s^2-4}{(s+1)(s-3)} = \frac{A(s-2)}{s+1} + B + \frac{C(s-2)}{s-3}$$

$$\frac{2(2)^2-4}{(2+1)(2-3)} = 0+B+0 \therefore \frac{-4}{3} = B$$

Multiply both sides by $(s-3)$, and substitute $S \to 3$

$$\underset{s\to 3}{Lt}\frac{2s^2-4}{(s+1)(s-2)} = \frac{A(s-3)}{s+1} + \frac{B(s-3)}{s-2} + C$$

$$\frac{2(3)^2-4}{(3+1)(3-2)} = 0+0+C$$

$$\therefore \frac{7}{2} = C$$

$$\therefore L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = L^{-1}\left\{\frac{\frac{-1}{6}}{s+1}+\frac{\frac{-1}{3}}{s-2}+\frac{\frac{7}{2}}{s-3}\right\} = -\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$$

3. $Find\ L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1}+\frac{B}{(s-2)^3}+\frac{C}{(s-2)^2}+\frac{D}{s-2}$$

By using the above procedure, the values $A = \frac{-1}{3}$ and $B = -7$ are obtained. This method fails to find C and D values. ∴ Substitute any two values for S. Let us Consider that S=0 and S=1

$For\ S = 0, we\ get,\quad \frac{11}{8} = -\frac{1}{3}+\frac{7}{8}+\frac{C}{4}-\frac{D}{2}$

$$On\ simplifying, \quad \frac{20}{24} = \frac{C}{4} - \frac{D}{2} \quad ie., \quad 3C - 6D = 10$$

$$And\ For\ S = 1, we\ get, \quad \frac{21}{2} = -\frac{1}{6} + 7 + C - D$$

$$On\ simplifying, \quad \frac{22}{6} = C - D \implies 3C - 3D = 11$$

$$On\ solving\ the\ equations\ we\ get\ the\ values \quad C = 4\ and\ D = \frac{1}{3}$$

$$L^{-1}\left\{\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right\} = L^{-1}\left\{\frac{\frac{-1}{3}}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-3)^2} + \frac{\frac{1}{3}}{s-2}\right\}$$

$$= -\frac{1}{3}e^{-t} - \frac{7}{2}t^2e^{2t} + 4t\,e^{2t} + \frac{1}{3}e^{2t}$$

4. $Find\ \ L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$Multiply\ both\ sides\ by\ \ (s-1), and\ substitute\ \ S \to\ 1\ \ we\ get, A = 2$

$Put\ S = 0, then\ the\ value\ of\ C = 1$

$Put\ S = 2, and\ simplify\ then\ we\ get\ the\ value\ of\ B = -2$

$$\therefore L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = L^{-1}\left\{\frac{2}{(s-1)} + \frac{-2s+1}{(s^2+1)}\right\}$$

$$= 2\,L^{-1}\left\{\frac{1}{(s-1)}\right\} - 2\,L^{-1}\left\{\frac{s}{(s^2+1)}\right\} + L^{-1}\left\{\frac{1}{(s^2+1)}\right\}$$

$$= 2e^t - 2\ \cos t + \sin t$$

5. $Find\ L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$Multiplying\ on\ both\ sides\ by\ \ (s^2+2s+2)(s^2+2s+5)\ \ we\ get$

$s^2 + 2s + 3\ = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$

$s^2 + 2s + 3 = (A + C)s^3 + (2A + B + 2C + D)s^2 + (5A + 2B + 2C + 2D)s + 5B + 2D$

Comparing the coefficient of powers of s on both sides we get,

$A + C = 0$; $2A + B + 2C + D = 1$; $5A + 2B + 2C + 2D = 2$; $5B + 2D = 3$

Solving these equations we get $A = 0$, $B = \frac{1}{3}$, $C = 0$, $D = \frac{2}{3}$

$$\therefore L^{-1}\left\{\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}\right\}$$

$$= L^{-1}\left\{\frac{\frac{1}{3}}{(s^2 + 2s + 2)} + \frac{\frac{2}{3}}{(s^2 + 2s + 5)}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2 + 4}\right\}$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{2}{3}\frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

<u>3.6 Application of Laplace Transform:</u> (Simple Harmonic Motion)

The equation of simple harmonic motion is

$$\frac{d^2y}{dt^2} + \omega^2 x = 0$$

$x$is the distance displaced the body from its mean position; ω is a constant

If $x = x_0$ (maximum distance displaced) then the initial conditions are

$$At\ t = 0, \qquad x = x_0\ and\ \frac{dx}{dt} = 0$$

$$Laplace\ transform\ on\ equation\ (1)\ we\ get\ L\left\{\frac{d^2x}{dt^2}\right\} + \omega^2 L\{x\} = 0$$

$$s^2\, L\{x\} - S\{x\}_{t=0} + \omega^2\, L\{x\} = 0$$

$$By\ initial\ condition\ s^2 f(s) - s x_0 + \omega^2 f(s) = 0$$

$f(s)(s^2 + \omega^2) = sx_0$

$$f(s) = \frac{sx_0}{s^2 + \omega^2}$$

Taking inverse Laplace Transform

$$x = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{sx_0}{s^2 + \omega^2}\right\} = x_0 L^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} = x_0 \cos \omega t$$

ie., The equation $x = x_0 \cos \omega t$ describes simple harmonic motion.

UNIT IV: COMPLEX ANALYSIS

Complex variables- complex conjugate and modulus of a complex number-algebraic operations of complex numbers-function of a complex variable-analytic function-Cauchy-Riemann equation in polar form-line integral of a complex function-Cauchy integral theorem-Cauchy integral formula-Derivatives of an analytic function

4.1 Complex Number:

A complex number is defined as a number of the form $z = a + ib$, where $i = \sqrt{-1}$, a and b are real numbers. a is real part of z $ie.,[Re(z)]$ and b is imaginary part of z $ie.,[Im(z)]$

(i) The complex number is Zero when and only when $x = 0$ and $y = 0$.

(ii) Two complex numbers z_1 and z_2 will be equal if the real and imaginary parts of each are equal.

(iii) If $a = 0, then\ z = ib\ and\ the\ complex\ number\ is\ purely\ imaginary$

(iv) If $b = 0, then\ z = a\ and\ the\ complex\ number\ is\ real$

(v) The sum, difference, product and ratio of two complex numbers is always a complex number

(vi) Complex conjugate: The complex number $z = a - ib$ is called the complex conjugate of z it is denoted by z^*. The sum and product of a complex number and its conjugate are both real.

$$(z^*)^* = z, \quad zz^* = (a + ib)(a - ib) = a^2 + b^2(real)\ and\ |z| = \sqrt{a^2 + b^2}$$

(vii) Polar form: Let $a = r\ \cos\theta \quad and \quad b = r\sin\theta$

$\therefore z = rcos\theta\ + i\ rsin\theta\ \ then\ z = r\ (cos\theta + isin\theta) = r\ e^{i\theta}$ This the polar form of complex number. Where $r = \sqrt{a^2 + b^2}$.

4.2 Properties of Modulus:

1. The modulus of the sum of two complex numbers z_1 and z_2 can never exceed the sum of their individual moduli. ie., $|z_1 + z_2| \leq (|z_1| + |z_1|)$

Example:

$z_1 = 5 + 4i$and $z_2 = 3 + 2i \therefore z_1 + z_2 = 8 + 6i$, then $|z_1 + z_2|$ = $\sqrt{8^2 + 6^2}$ =10

$|z_1|$ =$\sqrt{5^2 + 4^2} = 6.403$ and $|z_2|$ =$\sqrt{3^2 + 2^2} = 3.605$, $\therefore |z_1| + |z_1|$ =10.008

ie $|z_1 + z_2| < (|z_1| + |z_1|)$

2. The modulus of the difference of two complex numbers z_1 and z_2 can never be less than the difference of their individual moduli. ie., $|z_1 - z_2| \geq (|z_1| + |z_1|)$

Example:

$z_1 = 5 + 4i$and $z_2 = 3 + 2i \therefore z_1 - z_2 = 2 + 2i$, then

$|z_1 - z_2|$ = $\sqrt{2^2 - 2^2}$ =2.828; $|z_1|$ =$\sqrt{5^2 + 4^2} = 6.403$&$|z_2|$ =$\sqrt{3^2 + 2^2} = 3.605$,

$\therefore |z_1| - |z_1|$ =2.798 ie.,$|z_1 + z_2| > (|z_1| - |z_1|)$

3. The modulus of the product of two complex numbers z_1 and z_2 is the product of their individual moduli. ie., $|z_1 z_2| = (|z_1||z_1|)$

Example:

$z_1 = 5 + 4i$and $z_2 = 3 + 2i \therefore z_1 z_2 = 7 + 22i$, then

$|z_1 z_2|$ = $\sqrt{7^2 - 22^2}$ =23.08 ;$|z_1|$ =$\sqrt{5^2 + 4^2} = 6.403$&$|z_2|$ =$\sqrt{3^2 + 2^2} = 3.605$, $\therefore |z_1||z_1|$ =23.08 ie., $|z_1 z_2| = |z_1||z_1|$

4. The modulus of the quotient (division) of two complex numbers z_1 and z_2 is the quotient (division) of their individual moduli. ie., $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_1|}$

Example:

$$z_1 = 5 + 4i \;\; and \;\; z_2 = 3 + 2i \;\; \therefore \frac{z_1}{z_2} = \left(\frac{23}{13}\right) + i\left(\frac{2}{13}\right)$$

$$then \;\; \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{23}{13}\right)^2 + \left(\frac{2}{13}\right)^2} = 1.78$$

$|z_1|$ =$\sqrt{5^2 + 4^2} = 6.403$&$|z_2|$ =$\sqrt{3^2 + 2^2} = 3.605$

$$\therefore \frac{|z_1|}{|z_2|} = \frac{6.403}{3.605} = 1.78 ie., \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

<u>4.3 Algebraic Operations of Complex numbers</u>

1. Addition:

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined as $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

$$= (x_1 + x_2, \quad y_1 + y_2)$$

2. Subtraction:

Subtraction of a complex numbers $z_2 = (x_2, y_2)$ from $z_1 = (x_1, y_1)$ is defined as $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

$$= (x_1 - x_2, \quad y_1 - y_2)$$

3. Multiplication:

Multiplication of two complex numbers $z_1 = (x_1, y_1)$ &$z_2 = (x_2, y_2)$ is defined as $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$$= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1)$$

4. Division:

Division of a complex numbers $z_1 = (x_1, y_1)$ by $z_2 = (x_2, y_2)$ and is defined as

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

<u>4.4 Variable and functions:</u>

A symbol z, which can stand for any one of a set of complex numbers is called a complex variable. If for each value of the complex variable $(z = x + iy)$ in a certain region R, we have one or more values of $(\omega = u + iv)$, then ω is known as a complexfunction of z $ie.,\ \omega = f(z)$. The variable z is called an independent variable, ω is a dependent variable. The value of a function at $z = a$is $f(a)$.

$$\therefore f(z) = u(x, y) + iv(x, y)$$

where$u(x, y)$ is real part and $v(x, y)$ is imaginary part.

<u>Example</u>:

(i) $Z = 2i, then\ f(z) = z^2$ will be $f(2i) = (2i)^2 = -4$

(ii) if $z = x + iy, then\ f(z) = z^2$ will be $f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy$

Ie.,$f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = x^2 - y^2$ is real part and

$v(x, y) = 2xy$ is imaginary part.

(iii) $F(z) = |z|^2, \quad Z = x + iy, \quad \therefore |Z| = \sqrt{x^2 + y^2} \quad ie.,\ f(z) = x^2 + y^2$

$\therefore u = x^2 + y^2$and $v = 0$, the function is real.

4.5 Single and Multi-valued function:

If for each z value, there is only one value of ω, then ω is said to be a single valued function of z. Otherwise, ω is a Multi-valued function of z.

Example:

$\omega = \frac{1}{z}$ and $\omega = z^2$ are single valued functions of z. But $\omega = \sqrt{z}$ is a milti-valued function of z, it possesses two values $(\pm z)$ for each z except at $z = 0$.

4.6 Analytic function:

A function $f(z)$ which is single valued and differentiable with respect to z at all points of a region R is said to be an analytic function or regular function of z in that region.
The point at which an analytic function is not differentiable is known as a singular point of the function.

4.7 Cauchy-Riemann Conditions:

The necessary condition that $\omega = f(z) = u(x, y) + iv(x, y)$ be analytic in a region is that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If the partial derivatives are continuous in the region then Cauchy-Riemann equations are sufficient condition that $f(z)$ is analytic in the region.
The real $u(x, y)$ and imaginary $v(x, y)$ parts of an analytic function $f(z)$ are also known as conjugate functions. If one is given then other can be found so that $f(z) = u + iv$ is analytic.

Proof:

If $\omega = f(z)$ be single valued function of the variable $z = x + iy$, then the derivative of $\omega = f(z)$ is defined as

$$\frac{d\omega}{dz} = f^{'}(z) = \underset{\delta z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z}$$

provided that the limit exists and is the same for all the different paths along which $\delta z \to 0$.

(i) Necessary condition: Let δu and δv be increments of u and v respectively corresponding to increments δx and δy of x and y

$$\therefore f(z+\delta z) = (u+\delta u) + i(v+\delta v)$$

Now,

$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{(u+\delta u)+i(v+\delta v)-(u+iv)}{\delta z} = \frac{\delta u + i\delta v}{\delta z} = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}$$

$$f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z+\delta z)-f(z)}{\delta z} = \underset{\delta z \to 0}{Lt} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right)$$

Since δz approaches zero , first assume δz to be wholly real and then wholly imaginary.

Case 1 When δz is wholly real, then $\delta y = 0$; $\delta z = \delta x$

$$\therefore f'(z) = \underset{\delta x \to 0}{Lt} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Case 2 When δz is wholly imaginary, then $\delta x = 0$; $\delta z = i\delta y$

$$\therefore f'(z) = \underset{\delta y \to 0}{Lt} \left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$f'(z)$exists only if both cases are equal, then we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts from both sides we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ; \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

So the necessary condition for the existence of $f'(z)$ is that the CR-equations are to be satisfied.

(i) Sufficient Condition: Let $f(z)$ be a single valued function having partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of R and the CR-equations be also satisfied.

By Taylor's theorem for function of two variables, we have

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \cdots + i[v(x,y) + (\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y) + \cdots]$$

$$= f(z) + (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\delta x + (\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\delta y$$

Disregarding the terms beyond the first power of δx, δy

$$f(z+\delta z) - f(z) = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\delta x + (\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\delta y$$

Using CR- equation rewrite the above equation as

$$f(z+\delta z) - f(z) = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\delta x + (-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x})\delta y$$

Therefore $f^{'}(z)$ the derivative exists and $f(z)$ is analytic in the region

4.8 Polar form of CR - equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \ \ and \ \frac{\partial u}{\partial \theta} = -r\left(\frac{\partial v}{\partial r}\right)$$

$$z = x + iy = r(cos\theta + i\ sin\theta) = re^{i\theta}$$

$$f(z) = u + iv = f(re^{i\theta})$$

Harmonic: To prove u and v are harmonic, Differentiate both side of CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (4.1)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (4.2)$$

Differentiating on both sides of (1) with respect to x we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad (4.3)$$

Differentiating on both sides of (2) with respect to y we get,

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad (4.4)$$

$Comparing\ (3)\ and\ (4)\ we\ get,$ $\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$

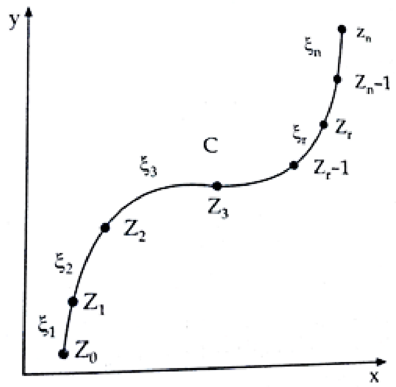
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (4.5)$$

Similarly we can get $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (4.6)$$

The second partial derivatives of u and v are continuous and harmonic. The equations (4.5) and (4.6) are called as Laplace equations.

4.9 Line integral of a complex Function:

Let C be a sooth curve with end points $z_0$and z_n. And let $z_1, z_2, \cdots z_r \cdots z_{n-1}$ are intermediate points which divide the curve C into n arcs $z_0 z_1$, $z_1 z_2, \cdots\cdots z_{r-1} z_r$, $\cdots\cdots z_{n-1} z_n$ as shown in figure



$\xi_1, \quad \xi_2, \cdots\cdots \xi_r, \quad \cdots\cdots \xi_n$ are the points lies on the corresponding arcs . Then we make the summation

$$S_n = \sum_{r=1}^{n} \xi_r \, \Delta z_r$$

Where $\Delta z_r = z_r - z_{r-1}$

when the curve divided into smaller and smaller

$n \to \infty$ then $|\Delta z_r| \to 0$ Then the summation S_n is known as the line integral of complex function $f(z)$ and is expressed as

$$\lim_{n\to\infty} S_n = \int_C f(z)dz$$

4.10 Cauchy's integral theorem:

Statement:

If a function $f(z)$ is analytic and $f'(z)$ is continuous at every point inside and on a simple closed curve C, then $\int_C f(z)dz = 0$.

Proof :

Let the region enclosed by the curve R and

$f(z) = u(x,y) + iv(x,y)$with$z = x + iy \Longrightarrow \quad dz = dx + idy$

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$\int_C (udx - vdy) + i\int_C (v\,dx + udy) \qquad (4.7)$$

Since $f^{'}(z)$ *is continuous* $, \frac{\partial u}{\partial x}\ ,\ \frac{\partial u}{\partial y}\ ,\ \frac{\partial v}{\partial x}\ ,\ \frac{\partial v}{\partial y}$ *are also continuous in* R

By applying Green's theorem,

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy \quad in\ each\ integral$$

we obtain $$\int_C (udx - vdy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dx\,dy \qquad (4.8)$$

$$\int_C (vdx + udy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dx\ \,dy \qquad (4.9)$$

Substituting (4.8) and (4.9) in (4.7) we get,

$$\int_C f(z)dz = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy + \mathrm{i}\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy \qquad (4.10)$$

Since $f(z)$ is analytic, u and v satisfy CR- equations so the integrands of the two integral in right hand side of equation (4.10) vanishes and we get,

$$\int_C f(z)dz = 0$$

Hence, proved the theorem.

<u>4.11 Properties of Line integral:</u>

1. $\int_C [f_1(z) + f_2(z)]\,dz = \int_C f_1(z)dz + \int_C f_2\,(z)\,dz$
2. $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
3. $\int_C f(z)dz = -\int_{-C} f(z)dz$
4. $\int_C k\,f(z)dz = k\int_C f(z)dz$

5. $\int_{z_1}^{z_2} f(z)dz = [F(z)]_{z_1}^{z_2} = F(z_2) - F(z_1)$

4.12 Cauchys Integral Formula:

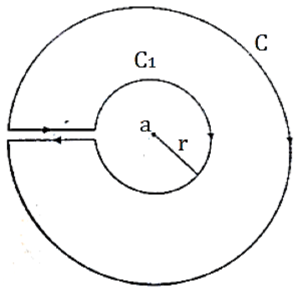
Statement:

If a function $f(z)$ is analytic within and on a closed curve C and if a is any point inside c then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Proof :

Let us consider the function $\frac{f(z)}{z-a}$ which is analytic at all points inside C, except at $z = a$. With point a as center and radius r, draw a small circle C_1 lying completely within C. (Figure)



Since $\frac{f(z)}{z-a}$ is analytic in the region between $C$and C_1, we have by Cauchy's theorem

Since for any point C_1 , $z - a = re^{i\theta} \Longrightarrow dz = ire^{i\theta}\,d\theta$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}}\,ire^{i\theta}\,d\theta = i\int_{C_1} f(a+re^{i\theta})\,d\theta$$

In the limit C_1 shrinks to point a ie., as$r \to 0$ $the\ integral\ approaches$

$$i\int_{C_1} f(a)\,d\theta = i\,f(a)\int_0^{2\pi} d\theta = 2\pi i\,f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

This is Cauchy's integral formula.

4.13 Derivatives of an analytic function:

When $f(z)$is an analytic function in a domain$D$, then its derivatives of all ordes exist and they also analytic function in D. The values of derivatives at any point z_0 in D are as the following

$$f^{'}(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f^{''}(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

And in general the n^{th} derivative is

$$f^{n}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Where C is a closed contour traversed in the anti-clockwise in D surrounding the point $z = z_0$.

Problems:

1. Determine the modulus and the principal argument of the complex number $\frac{1+2i}{1-(1-i)^2}$

Solution:

$$\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-1-2i)} = \frac{1+2i}{1+2i} = 1$$

$$1 + i.0 = x+iy = |x+iy| = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = 1$$

$$\therefore Modulus = \left|\frac{1+2i}{1-(1-i)^2}\right| = 1$$

Principal argument is

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1}(0) = 0$$

2. Convert $12\angle-60°$ to rectangular form

Solution:

Given $r=12$and $\theta=-60$. Let $12\angle-60°=+iy$, but $x+iy=r(cos\theta+isin\theta)$

ie., $x=rcos\theta=12\cos(-60)=12*\frac{1}{2}=6$

$y=rsin\theta=12\sin(-60)=-12*\frac{\sqrt{3}}{2}=-6\sqrt{3}$

$\therefore 12\angle-60=6-i6\sqrt{3}$

3. *Express the following complex number in* $r(cos\theta+isin\theta)$ *form* $\frac{(1+2i)}{(1-3i)}$

Solution:

$\frac{(1+2i)}{(1-3i)}=\frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}=\frac{1-6+5i}{1+9}=\frac{-5+5i}{10}=-\frac{1}{2}+\frac{i}{2}$

ie., $r(cos\theta+isin\theta)=-\frac{1}{2}+\frac{i}{2}\Longrightarrow rcos\theta=-\frac{1}{2}\ ;\ rsin\theta=\frac{1}{2}$

Squaring and adding we get, $r^2=\frac{1}{2}\Longrightarrow r=\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}cos\theta=-\frac{1}{2}\Longrightarrow Cos\theta=-\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}sin\theta=\frac{1}{2}\Longrightarrow sin\theta=\frac{1}{\sqrt{2}}\quad\therefore\ \theta=\frac{3\pi}{4}$

$\frac{(1+2i)}{(1-3i)}=\frac{1}{\sqrt{2}}\left(cos\frac{3\pi}{4}+i\,sin\frac{3\pi}{4}\right)$

4. *If* $x+\frac{1}{x}=2cos\theta,$ *prove that* $2\cos r\theta=x^r+\frac{1}{x^r}$

Solution:

$x+\frac{1}{x}=2cos\theta\Longrightarrow x^2-2xcos\theta+1=0$

$\therefore x=\frac{2cos\theta\pm\sqrt{4cos^2\theta-4}}{2}=cos\theta\ \pm isin\theta$

$x^r=(cos\theta\ \pm isin\theta)^r=cosr\theta\ \pm isinr\theta$

$x^{-r}=(cos\theta\ \pm isin\theta)^{-r}=cosr\theta\ \mp isinr\theta$

$x^r+\frac{1}{x^r}=(cosr\theta\ \pm isinr\theta)+(cosr\theta\ \mp isinr\theta)=2cosr\theta$

5. $For\ a\ complex\ variable\ z, resolve\ \ ln(z)\ \ into\ real\ and\ imaginary\ parts.$

Solution:

$we\ have\ \ z = x + iy\ , using\ x = rcos\theta\ and\ y = r\ sin\theta$

$$z = rcos\theta + rsin\theta \Longrightarrow r(cos\theta + isin\theta) \Longrightarrow re^{i\theta}$$

Where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \ln(z) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln e^{i\theta} = \ln\sqrt{x^2 + y^2} + \mathrm{i}\,\theta$

$$= \frac{1}{2}\ln(x^2 + y^2) + \ i\ \tan^{-1}\left(\frac{y}{x}\right)$$

Real part is $\frac{1}{2}\ln(x^2 + y^2)$and the imaginary part is $\tan^{-1}\left(\frac{y}{x}\right)$

6. $Express\ the\ following\ into\ real\ and\ imaginary\ parts.$ $(i)\ \sqrt{5+4i}$ $(ii)\ \sqrt{i}$

(i) Let $\sqrt{5+4i}$ $= a + ib$

Squaring on both sides, $5 + 4i$ = $(a + ib)^2$ = $a^2 - b^2 + i\ 2ab$

Thus, $a^2 - b^2$= 5 ; $2ab$ = 4 $\Longrightarrow ab$ = 2 $\Longrightarrow a = \frac{2}{b}$

$\therefore \left(\frac{2}{b}\right)^2 - b^2 = 5 \Longrightarrow b^4 + 5b^2 - 4 = 0$

$$b^2 = \frac{-5 \pm \sqrt{5^2 - 4.1.(-4)}}{2} = \frac{-5 + \sqrt{41}}{2} = 0.702$$

$\therefore b = \sqrt{0.702} = 0.837 \quad and \quad a = \dfrac{2}{0.837} = 2.389$

Therefore we get $\sqrt{5+4i}$ = $2.389 + i\ 0.839$

(ii) Let $\sqrt{i}$ $= a + ib$

$Squaring\ on\ both\ sides, \quad we\ get,\ \ i\ =\ (a + ib)^2\ =\ a^2 - b^2 + i\ 2ab$

$Thus, \quad a^2 - b^2\ =\ 0 \Longrightarrow a^2 = b^2\ ; \quad 2ab\ =\ 1 \quad \Longrightarrow ab\ =\ \dfrac{1}{2} \Longrightarrow a = \dfrac{1}{2b}$

$\therefore \left(\frac{1}{2b}\right)^2 - b^2 = 0 \Longrightarrow \left(\frac{1}{2b}\right)^2 = b^2$

$$Square\ root\ on\ both\ sides\ we\ get,$$

$\frac{1}{2b} = b \Longrightarrow b^2 = \frac{1}{2}\ \ Then\ b = \pm\frac{1}{\sqrt{2}} \therefore a = \pm\frac{1}{\sqrt{2}}$

$ie.,\sqrt{i} = \pm\frac{1}{\sqrt{2}}\,(i+i)$

7. Determine the analytic function $f(z) = u + i\,v$ whose imaginary part is

$$v = 6xy - 5x + 3$$

Solution:

Given $v = 6xy - 5x + 3$

$\frac{\partial v}{\partial x} = 6y - 5;$ and $\frac{\partial v}{\partial y} = 6x$

$For\ u = u(x,y),\ \ we\ have\ \ du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$

Using CR equations, we get,

$du = \frac{\partial v}{\partial y}\,dx - \frac{\partial v}{\partial x}\,dy = 6x\,dx - (6y-5)dy$

On integrating, we get

$u = \int 6x\,dx - \int (6y-5)\,dy + C$

$= 6\,\frac{x^2}{2} - 6\frac{y^2}{2} + 5y + C$

$$u = 3x^2 - 3y^2 + 5y + C$$

$\therefore f(z) = u + i\,v = (3x^2 - 3y^2 + 5y + C) + i\,(6xy - 5x + 3)$

8. Show that$e^x(cosy + isiny)$ is an analytic function, Find its derivative

Solution:

$Let\ f(z) = e^x(cosy + isiny) = u + iv \Longrightarrow e^x cosy = u;\ e^x siny = v$

$\therefore \frac{\partial u}{\partial x} = e^x cosy\,;\ \ \frac{\partial v}{\partial x} = e^x siny\ \ and\ \frac{\partial u}{\partial y} = -e^x siny\,;\ \ \frac{\partial v}{\partial y} = e^x cosy$

$it\ satisfies\ CR\ equations\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

∴ The given function $f(z) = e^x(cosy + isiny)$ is analytic.

The derivative

$f'^{(z)} = \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x}$

$= e^x cosy + i\ e^x siny = e^x(cosy + isiny) = e^x e^{iy} = e^{x+iy} = e^z$

9. Using Cauchy-Riemann condition show that $W = sinz$ is analytic

$W = sinz \Longrightarrow \sin(x + iy) = sinx\ coshy + i\ cosx\ sinhy$

$ie., u = sinx\ coshy \Longrightarrow \frac{\partial u}{\partial x} = cosx\ coshy\ \ and\ \ \frac{\partial u}{\partial y} = sinx\ sinhy$

$v = cosx\ sinhy \Longrightarrow \frac{\partial v}{\partial y} = cosx\ coshy \quad and \quad \frac{\partial v}{\partial x} = -sinx\ sinhy$

$it\ satisfies\ CR\ equations\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore W = sinz\ \ is\ analytic.$

10. Which of the following are analytic functions of complex variable, $z = x + iy$

(i) $|z|$ (ii) z^{-1} (iii) e^{sinz}

Solution:

(i) $we\ have\ |z| = |x + iy| = \sqrt{x^2 + y^2}$

$For\ f(z) = |z|, \quad then\ \ |z| = u + iv = \sqrt{x^2 + y^2}$

$ie., u = \sqrt{x^2 + y^2}\ \ and\ v = 0 \therefore \frac{\partial v}{\partial x}\ and\ \frac{\partial v}{\partial y}\ \ do\ not\ exists.$

And the function $f(z) = |z|$ is not analytic.

(ii)

$$f(z) = z^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{(x - iy)}{(x + iy)(x - iy)} = \frac{(x - iy)}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

$ie.,\quad u = \frac{x}{x^2 + y^2}\ \ and\ v = \frac{y}{x^2 + y^2}$

Taking derivatives we get, $u = x\,(x^2+y^2)^{-1}$

$$\therefore\ \frac{\partial u}{\partial x} = x.(-1)(x^2+y^2)^{-2}2x + (x^2+y^2)^{-1}.1$$

$$=.\frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{(x^2+y^2)} = \frac{-2x^2+x^2+y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$Similarly\ we\ get,\ \ \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}\ ;\ \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}\ ;and\ \frac{\partial v}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

CR-equation satisfied, therefore $f(z) = \frac{1}{z^{-1}}$ is analytic.

$(iii)\quad f(z) = e^{sinz} \Longrightarrow e^{\sin(x+iy)} = e^{sinx\ cos(iy)+cosx\ \sin(iy)} = e^{sinx\ coshy+i\ cosx\ sinhy}$

$= e^{sinx\ coshy}e^{i\ cosx\ sinhy} \quad = e^{sinx\ coshy}[cos(cosx\ sinhy)+i\ sin(cosx\ sinhy)]$

For $f(z) = u + i\,v = e^{sinx\ coshy}\cos(cosx\ sinhy) + i\ e^{sinx\ coshy}\sin(cosx\ sinhy)$

ie.,$u = e^{sinx\ coshy}\cos(cosx\ sinhy)$ and $v = e^{sinx\ coshy}\sin(cosx\ sinhy)$

Taking derivatives we get

$$\frac{\partial u}{\partial x} = e^{sinx\ coshy}[sinx\ sinhy\cos(cosx\ sinhy) + cosx\ coshy\cos(cosx\ sinhy)]$$

$$\frac{\partial u}{\partial y} = e^{sinx\ coshy}[sinx\ sinhy\cos(cosx\ sinhy) - cosx\ coshy\sin(cosx\ sinhy)]$$

$$\frac{\partial v}{\partial x} = -e^{sinx\ coshy}[sinx\ sinhy\cos(cosx\ sinhy) - cosx\ coshy\sin(cosx\ sinhy)]$$

$$\frac{\partial v}{\partial y} = e^{sinx\ coshy}[sinx\ sinhy\cos(cosx\ sinhy) + cosx\ coshy\cos(cosx\ sinhy)]$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\quad and\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

CR equations satisfied and the given function $f(z) = e^{sin\ z}$ is analytic.

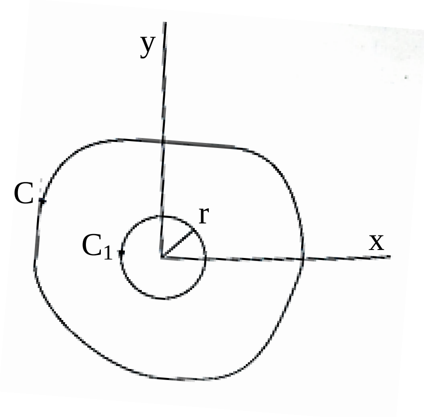
11. For a simple closed curve C, evaluate the following integral with the help of Cauchy integral theorem

$$\oint_C \frac{dz}{z}$$

.

Solution:

Given that $f(z) = \frac{1}{z}$ is analytic for all values of z except for $= 0$, and C is closed encloses the origin. Draw an arc C_1 of radius r with origin as Center.



$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z}$$

On C_1 we have, $z = r\, e^{\,i\theta}$ so that $dz = i\, r\, e^{\,i\theta}$ then

$$\frac{dz}{z} = \frac{i\, r\, e^{\,i\theta}}{r\, e^{\,i\theta}} = i\theta$$

$$\therefore \oint_{C_1} \frac{dz}{z} = \int_0^{2\pi} i\theta = 2\pi i \qquad \therefore \oint_C \frac{dz}{z} = 2\pi i$$

If C does not enclose the origin then $f(z) = \frac{1}{z}$ is analytic for all values of z

$$ie., \oint_C \frac{dz}{z} = 0$$

$$\therefore \oint_C \frac{dz}{z} = \begin{cases} 0, & When\ C\ does\ not\ enclose\ the\ origin \\ 2\pi i, & when\ C\ encloses\ the\ origin \end{cases}$$

12. Evaluate the following integrals

$(i) \int_1^3 (z-1)^2\ dz \qquad (ii) \int_0^{\pi i} z\, cos^2 z\ dz$

Solution:

$$(i) \int_i^3 (z-1)^2\ dz = \left[\frac{(z-1)^3}{3}\right]_i^3 = \frac{2^3}{3} - \frac{(i-1)^3}{3} = \frac{8 - \{i^3 - 1 - 3i\ (i-1)\}}{3} = \frac{6-2i}{3}$$

$$(ii)\ Let \quad I = \int_0^{\pi i} z\, cos^2 z\ dz$$

$Put\ z^2 = t, \quad then\ \ 2z\, dz = dt\,;$

$the\ limits\ \ when\ \ z = 0, t = 0 \quad and \quad z = \pi i, \quad t = (\pi i)^2 = -\pi^2$

$$\therefore I = \int_0^{-\pi^2} \frac{1}{2} \cos t\ dt = \frac{1}{2} [\sin t]_0^{-\pi^2} = -\frac{1}{2} \sin \pi^2$$

13. $Evaluate\ the\ integral\ \oint_C \frac{dz}{z^2 + z}$

$where\ C\ represents\ a\ circle\ defined\ by\ R = |z| > 1$

Solution:

The poles of the integrand obtained by putting denominator equal to Zero

$ie., z^2 + z = 0\ ; z(z+1) = 0$

$It\ gives\ two\ poles, \quad z = 0 \quad and\ \ z = -1.\ \ as\ \ R = |z| > 1$

Both poles lie within the contour. On eliminating these poles by drawing circles C_1 and C_2 of small radii and making cross-cuts to form simply connected region, we get

$$\oint_C \frac{dz}{z^2 + z} = \oint_{C_1} \frac{dz}{z^2 + z} + \oint_{C_2} \frac{dz}{z^2 + z}$$

$$= \oint_{C_1} \frac{dz/(z+1)}{z} + \oint_{C_2} \frac{(dz/z)}{z+1}$$

Using Cauchys integral formula, we get

$$\oint_C \frac{dz}{z^2 + z} = 2\pi i \left[\frac{1}{z+1}\right]_{z=0} + 2\pi i \left[\frac{1}{z}\right]_{z=-1} = 2\pi i - 2\pi i = 0$$

14. Evaluate the integrals

$$(i)\ \oint_C \frac{sin\, z}{z^2}\, dz \qquad (ii)\ \oint_C \frac{e^{az}}{(z - z_0)^3}\, dz$$

where$C$ represents a circle defined by $R = |z| > z_0$

(i) There is a pole of degree two at $z = 0$, we have

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

Given $f(z) = \sin z$, $n = 1\ and\ z_0 = 0\ and\ f'(z) = \cos z\ \ and\ f'(0) = 1$

$$\therefore \oint_C \frac{\sin z}{z^2} dz = 2\pi i$$

(ii) There is a pole of degree three at $z = z_0$, we have

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

Given $f(z) = e^{az},\ n = 2$

$Then\ \ f^{'}(z) = ae^{az}\ and\ f^{''}(z) = a^2 e^{az}\ \ when\ z\ \rightarrow z_0\ , f^{''}(z_0) = a^2 e^{az_0}$

Substituting these values we get

$$\oint_C \frac{e^{az}}{(z-z_0)^3} dz = \frac{2\pi i}{2!} f^2(z_0) = \pi i a^2 e^{az_0}$$

15. Let $f(z) = u + iv$ be an analytic function. If $u = x^3 + 3x^2y - 3xy^2 - y^3$, find out v.

Solution:

$$u = x^3 + 3x^2y - 3xy^2 - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy - 3y^2; \qquad \frac{\partial^2 u}{\partial x^2} = 6x + 6y$$

$$\frac{\partial u}{\partial y} = 3x^2 - 6xy - 3y^2; \qquad \frac{\partial^2 u}{\partial y^2} = -6x - 6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6y - 6x - 6y = 0$$

therefore$u$ can be a part of analytic function $f(z) = u + iv$

$For\ v = v(x, y), we\ have\ \ dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

The using CR-equation, we get

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -(3x^2 - 6xy - 3y^2)dx + (3x^2 + 6xy - 3y^2)dy$$

$$v = \int (6xy + 3y^2 - 3x^2)dx + \int (3x^2 + 6xy - 3y^2)dy + C$$

where$C$ is constant of integration

$$v = 3x^2y + 3y^2x - x^3 - y^3 + c$$

Hence the function is

$f(z) = x^3 + 3x^2y - 3xy^2 - y^3) + i\,(3x^2y + 3y^2x - x^3 - y^3 + c)$

16. $Let\ \ f(z)\ = u + iv\ \ be\ an\ analytic\ function. If\ v$

$$= x^3 - 3xy^2 + 3x^2 - 3y^2, find\ u.$$

Solution:

$v = x^3 - 3xy^2 + 3x^2 - 3y^2$

$\frac{\partial v}{\partial x} = 3x^2 - 3y^2 + 6x;\qquad \frac{\partial^2 u}{\partial x^2} = 6x + 6$

$\frac{\partial v}{\partial y} = -6xy - 6y;\qquad \frac{\partial^2 v}{\partial y^2} = -6x - 6$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6x + 6 - 6x - 6\ = 0$

therefore$v$ can be a part of analytic function $f(z)\ =\ u\ +\ iv$

$For\ u\ =\ u(x, y), we\ have\ \ du = \frac{\partial u}{\partial x}\ dx + \frac{\partial u}{\partial y}\ dy$

The using CR-equation, we get

$du = \frac{\partial v}{\partial y}\ dx - \frac{\partial v}{\partial x}\ dy = (-6xy - 6y)dx +\ (3x^2 + 6x - 3y^2)dy$

$u = \int(-6xy - 6y)dx + \int\ (3x^2 + 6x - 3y^2)dy + C$

where$C$ is constant of integration

$$u = -3x^2y - 6xy + y^3 + c$$

Hence the function is

$f(z) = (y^3 - 3x^2y - 6xy + c) + i\ (x^3 - 3xy^2 + 3x^2 - 3y^2)$

UNIT V: GROUP THEORY

Concept of a group-Group multiplication table of order 2, 3, 4 groups- Group symmetry of equilateral triangle- Group symmetry of a square-permutation group-conjugate elements-representation through similarity transformation-reducible and irreducible representation-SU(2) group-SO(2) group.

5.1 Introduction:

Group theory is a branch of mathematics which can be applied to any set of elements which can be applied to any set of elements which obey the necessary conditions to be called a group. The symmetry operation can be considered as elements.

Group is a set of elements A, B, C, …..and satisfies the following conditions

(i) Closure property: The product of any two elements in the group and the square of each element must be an element in the group. ie., if $A, B \in G$; then$A \cdot B$, $B \cdot A \in G$; $A^2, B^2 \in G$

(ii) The associative law: Theassociative law of multiplication must hold. ie., if $A, B, C \in G$; then $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

(iii) Existence of Identity: One element in the group must commute with all others and leave them unchanged. This element is unit element or identity element E. ie., $A, E \in G$ Then $A \cdot E = E \cdot A = A$

(iv) Existence of Identity: Every element must have a reciprocal, which is also an element of the group.$A^{-1} = B \in G$then $A \cdot A^{-1} = A^{-1} \cdot A = E$

5.2 Finite group:

The finite group contains a finite number of group elements. The number of elements in a group is called its order and is represented as h. A set of covering operations of a symmetrical object is an example of a finite group. Covering operation means a rotation, reflection or inversion which would bring the object into a form indistinguishable from the original one.

5.3 Abelian group:

If the multiplication of two elements in a group is commutative then the group is abelian. ie., $AB = BA\ for\ all\ A\ and\ B\ in\ the\ group$. Abelian group of infinite order is set of all positive and negative integers including Zero.Ordinary addition serves as the group multiplication operation. Zero serves as the unit element and $-n$ is the inverse of n. The set is closed and the associated law is obeyed.

5.4 Non-Obelian group:

Finite order of this group is the set of all $n \times n$ matrices with non vanishing determinants. Group multiplication operation is the matrix multiplication. Unit element is $n \times n$ unit matrix. The inverse matrix of each matrix element is inverse element.

5.5 Cyclic group:

If A is an element of a group G all integral powers of A such as A^2, A^3, … must also be in G. If G is a finite group $A^h = E$ where h is the order of the group G. In general the cyclic group of order h is defined as an element A and all of its powers up to $A^h = E$. All cyclic groups must be abelian. Example of standard triangle, the sequence period of D is D, $D^2 = F$, $D^3 = DF = E$. Therefore order of D=3, and D, F, E form a cyclic subgroup of our entire group of order 6. Ie., all the elements of a group can be generated from one group.

5.6 Group Multiplication Table:

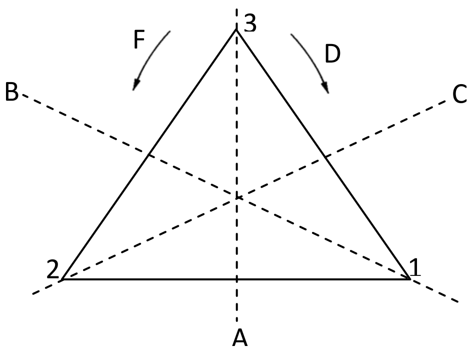
Multiplication Table consists of h rows and $h$columns. Each column is labeled with a group element and so in row. Each entry is the product of the element labeling the row times the element labeling the column ie$AB = D \neq BA$ Example with 6 elements

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

If we take the elements to be the following 6 matrices and if ordinary multiplication is used as the group multiplication operation

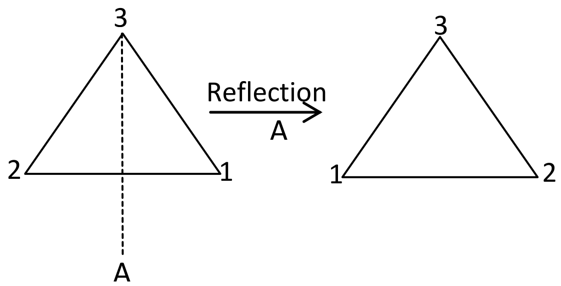
$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \qquad D = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

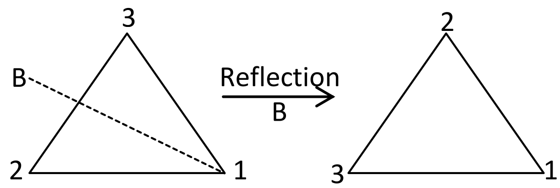
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \qquad F = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

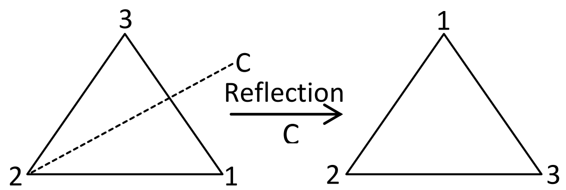


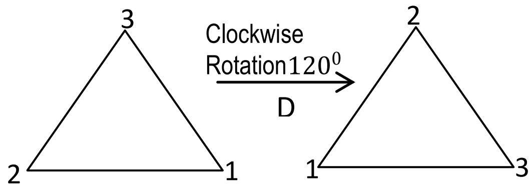
The same multiplication table can be obtained by group symmetry of equilateral triangle.The symmetry operations of an equilateral triangle with corresponding symbols are as follow.

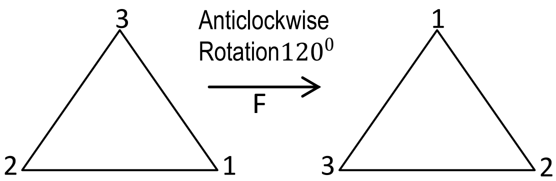
The elements A, B and C are getting by rotation of an angle π or reflection about the axes shown. D is obtained by a clockwise rotation of $\frac{2\pi}{3}$ $or\ 120°$ angle in the plane of the triangle and F is attained by a counter clockwise rotation through an angle of $\frac{2\pi}{3}$ $or\ 120°$. The product AB means the operations obtained by performing B first and then A.











Groups of order 1: The group contains only the identity element E.

Groups of order 2: The group consists the elements A, $A^2 = E$. This is is an obelian group. A might represent reflection , inversion or interchange of two identical particles.

	E	A
E	E	A
A	A	E

Groups of order 3: The elements are A,B, andE. Here $A^2 = B \neq E$

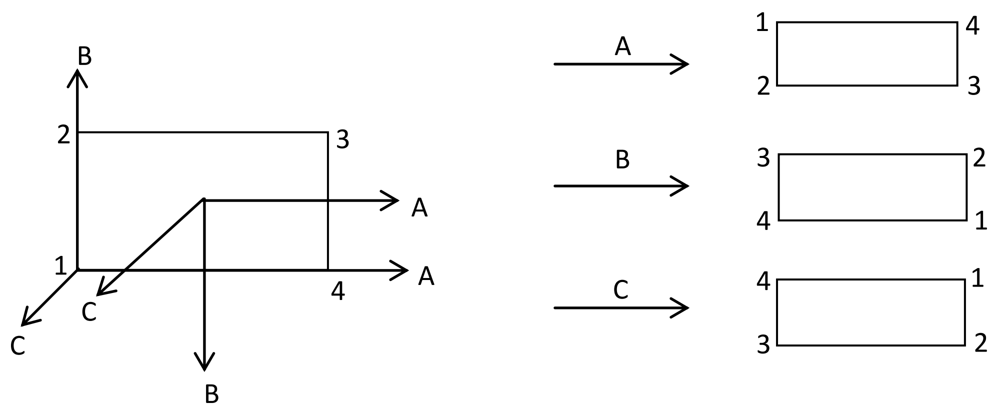
	E	A	B
E	E	A	B
A	A	B	E
B	B	E	A

A, $A^2 = B$, $A^3 = A A^2 = AB = E$ forms a cyclic group.

Groups of order 4: Two possibilities of group multiplication.

(i) the cyclic group → four fold rotation about an axis. A, A^2, A^3, $A^4 = E$.

(ii) Vierergruppe(A,B,C,E) → rotational symmetry group of a rectangular solid if A, B, C are taken to be rotation by π angle about the 3 orthogonal axes.



Both are abelian groups we can get subgroups of order 2.

The group multiplication table for four elements is

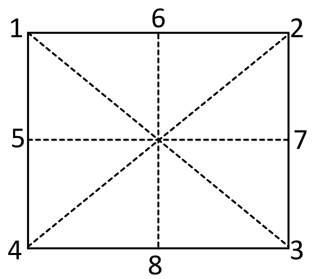
	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

5.7 Groups of prime order:

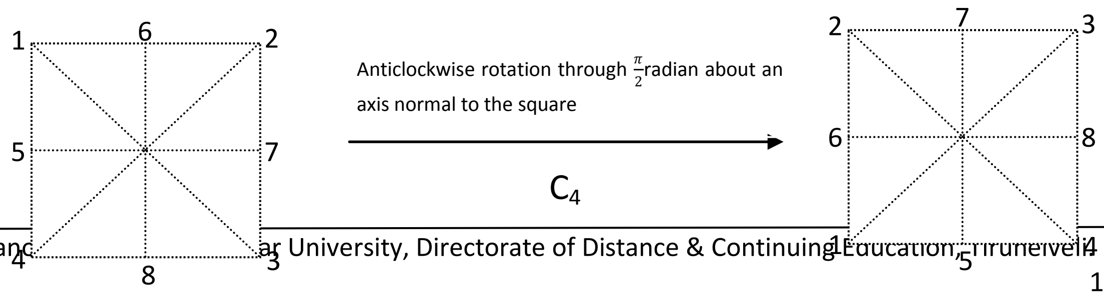
These are cyclic abelian groups. The period of some element would have to appear as a subgroup whose order was a divisor of a prime number. There can be only single group of order 1, 2, 3, 5, 7, 11, 13, etc.

5.8 Group Symmetry of a square:

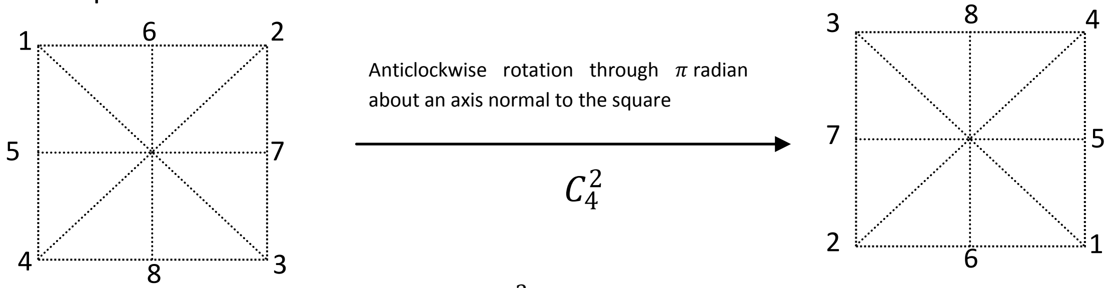
Consider a square ABCD with M, N, O, P as mid points of sides as shown in the figure. The covering operations of a square form D_4 containing eight elements. They are $\{E,\ C_4,\ C_4^2,$ $C43,\ mx, my, \sigma x, \sigma y$. The transformations are given below.



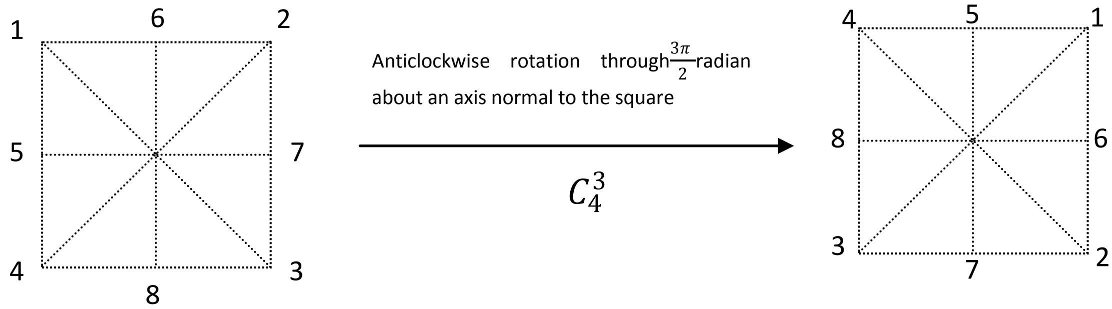
1. E→their no Transformation
2. C_4 → anti-clockwise rotation through $\frac{\pi}{2}$ radian or 90° angle about anaxis normal to the square



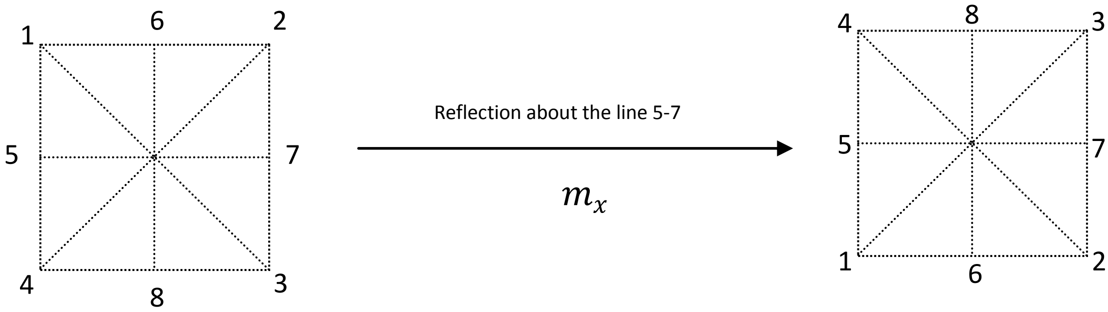
3. C_4^2 → anti-clockwise rotation through π radian or 180^o angle about anaxis normal to the square



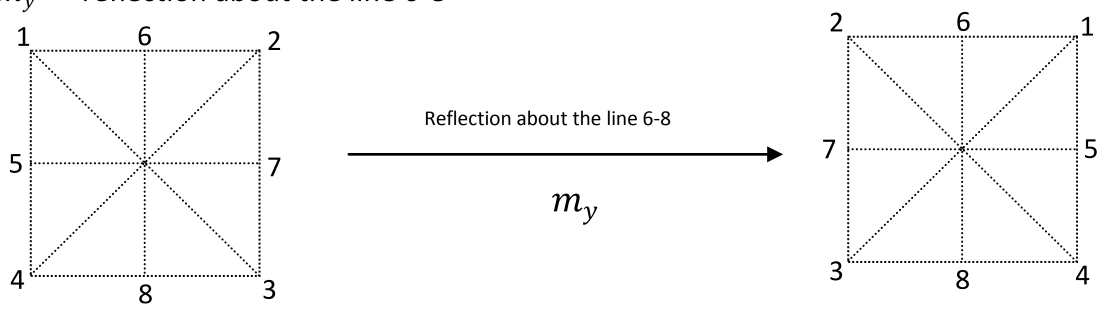
4. C_4^3 → anti-clockwise rotation through $\frac{3\pi}{2}$ radian or 180^o angle about anaxis normal to the square



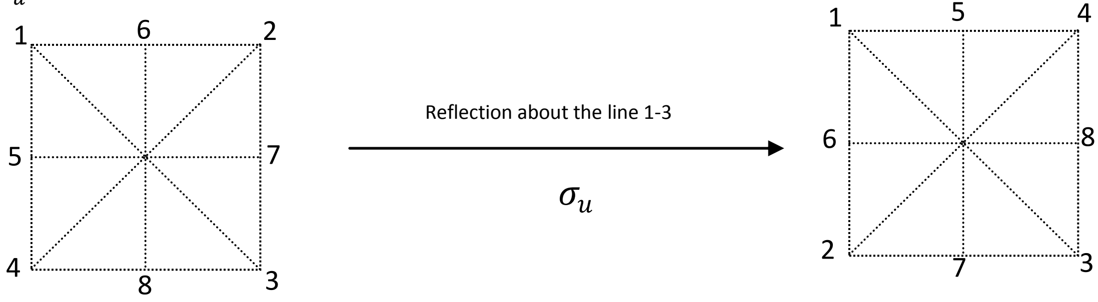
5. m_x → reflection about the line 5-7



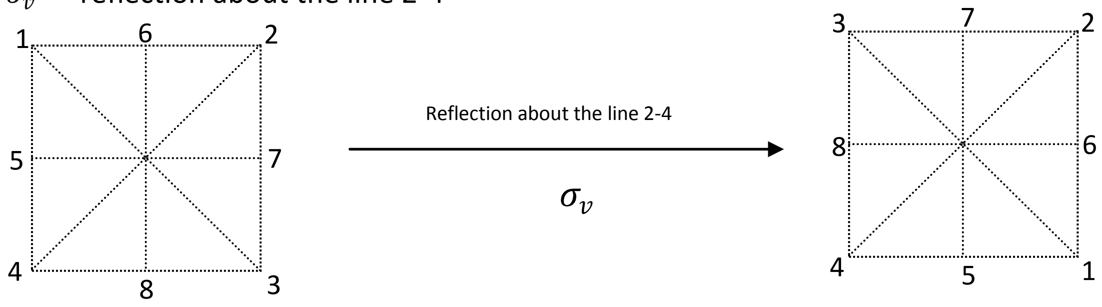
6. m_y → reflection about the line 6-8



7. σ_u → reflection about the line 1-3



8. σ_v → reflection about the line 2-4



The combination of any two operations is equivalent to one of these operations. The group multiplication table is as follows

	E	C_4	C_4^2	C_4^3	m_x	m_y	σ_u	σ_v
E	E	C_4	C_4^2	C_4^3	m_x	m_y	σ_u	σ_v
C_4	C_4	C_4^2	C_4^3	E	σ_u	σ_v	m_y	m_x
C_4^2	C_4^2	C_4^3	E	C_4	m_y	m_x	σ_v	σ_u
C_4^3	C_4^3	E	C_4	C_4^2	σ_v	σ_u	m_x	m_y
m_x	m_x	σ_v	m_y	σ_u	E	C_4^2	C_4^3	C_4
m_y	m_y	σ_u	m_x	σ_v	C_4^2	E	C_4	C_4^3
σ_u	σ_u	m_x	σ_v	m_y	C_4	C_4^3	E	C_4^2
σ_v	σ_v	m_y	σ_u	m_x	C_4^3	C_4	C_4^2	E

5.9 Permutation groups: (of factorial order)

The permutation of a set is defined as one-to-one mapping of a finite set onto itself. if $\alpha_1,\alpha_2,\ldots\ldots..\alpha_n$ be an arrangement of the set of integers 1, 2,…….n, then A permutation can be specified by a symbol of one-to-one mapping of the finite set {1,2,3,…,n} onto itself such as

$$p = \begin{pmatrix} 1 & 2 & 3\cdots\cdots\cdots & n \\ \alpha_1 & \alpha_2 & \alpha_3\cdots\cdots\cdots & \alpha_n \end{pmatrix}$$

In the above symbol, the order of a column is normally immaterial so long the corresponding elements above and below in the column remain the same. ie., $\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$, $\begin{pmatrix} b & c & a \\ c & a & b \end{pmatrix}$ and $\begin{pmatrix} c & a & b \\ a & b & c \end{pmatrix}$ represent the same permutation

The number of elements of a finite set is the degree of the permutation. The permutation p of the set $S = \{1,2,3,\ldots,n\}$ means that by mapping p, the $(\alpha_1,\alpha_2,\ldots\ldots..\alpha_n)$are the images of $(1,2,3,\ldots\ldots n)$ and may be expressed as

$p(1) = \alpha_1$, $p(2) = \alpha_2$, $p(3) = \alpha_3$,……………, $p(n) = \alpha_n$

Example:

If S is not too large, it is feasible to describe a permutation by listing the elements $x \in S$ and the corresponding values $p(x)$.

For example, if $S = \{1,2,3,4,5\}$, then $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{bmatrix}$ is the permutation such that $p(1) = 3$, $p(2) = 5$, $p(3) = 4$, $p(4) = 1$, $p(5) = 2$.

If we start with any element $x \in S$ and apply p repeatedly to obtain $p(x)$, $p\big(p(x)\big)$, $p(p(p(x)))$, and so on, eventually we must return to x, and there are no repetitions along the way because p is one-to-one.

For the above example, we obtain $1 \to 3 \to 4 \to 1$, $\quad 2 \to 5 \to 2$

We express this result by writing $\quad p = (1,3,4)(2,5)$

where the cycle (1, 3, 4) is the permutation of S that maps 1 to 3, 3 to 4 and 4 to 1, leaving the remaining elements 2 and 5 fixed. Similarly, (2, 5) maps 2 to 5, 5 to 2, 1 to 1, 3 to 3 and 4 to 4. The product of (1, 3, 4) and (2, 5) is interpreted as a composition,with the right factor

(2, 5) applied first, as with composition of functions.In this case, the cycles are disjoint, so it makes no difference which mapping is applied first. The above analysis illustrates the fact that any permutation can be expressed as a product of disjoint cycles, and the cycle decomposition is unique.

If the set contains $n$elements , the set of permutations p will have $n!$ elements, for n distinct objects can be arranged or permuted in $n!$ ways. If $n = 3$, the number of permutations are 3! or 6.

The permutation in which the item in position 'i' is shifted to the position indicated in the lower line. Successive permutation forms the group multiplication operation. If by permutation, there is no change in the elements, the permutation is known as identity permutation E. On combining any two permutations by multiplication we get another permutation

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = D$$

To obtainAB, rearrange the order of column in B such that the first row of B becomes (2 1 3) identical with the second row of A (2 1 3) to get there by cancelled.

$$AC = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = F$$

$$A^2 = AA = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = E$$

Permutation multiplication is not commutative. But the Permutation multiplication is associative.

5.10 Conjugate element:

An element B is conjugate to A if $B = X A X^{-1}$ or $A = X B X^{-1}$ where X is some member of the group. If B and C conjugate to A, then they are conjugate to each other.

Proof: Let $B = X A X^{-1}$, and $C = Y A Y^{-1}$

From C we get, $A = Y^{-1} C Y$, substitute this in B, we get $B = X Y^{-1} C Y X^{-1}$; $B = X Y^{-1} C (X Y^{-1})^{-1}$ ie $B = Z C Z^{-1}$ ∴ B and C conjugate

5.11 Representation of a group:

Let $G = \{E, A, B, \cdots\cdots\}$ be a finite group of order g with E as the identity element. And let $T = \{T(E),\ T(A),\ T(B), \cdots\cdots\}$ be a collection of nonsingular square matrices all of them are having the same order with the property $T(A)T(B) = T(AB)$ie., if $AB = C$, in the group G, then $T(A)T(B) = T(C)$. The collection T of matrices is said to be a representation of the group G. The order of the matrices of T is called the dimension of the representation.

If the matrices of the set T are all distinct there will be one-to-one correspondence between the elements of the group G and the set T ie., the two groups G and T are isomorphic to each other and such a representation is true.

If the matrices of the set T are not all distinct then the groups G and T are homomorphic or isomorphic to each other and such a representation is an unfaithful representation of G.

The 2×2 matrices represented for elements E, A, B, C, D, F of group of order 6 is a faithful matrix representation. Another representation of the same group can be obtained by taking the determinant of each matrix $|T(A)| \cdot |T(B)| = |T(AB)|$. This operation reduces the matrix to ordinary numbers ± 1. Thus this representation consists of only two distinct matrices for six group elements and hence is unfaithful representation.$|T(E)| \cdot |T(A)| = |T(A)| \cdot |T(E)| = |T(A)|$Such that $|T(A)| \neq 0$. This matrix equation is satisfied only if $T(E) = E$, the unit matrix. Thus in any representation the identity element of the group must be represented by the unit matrix of appropriate order.

Similarity transformations leave the multiplication properties of matrices unchanged.
ie if we define $T^{'}(A) = S^{-1} T(A) S$

$$T^{'}(A) T^{'}(B) = S^{-1} T(A) S\, S^{-1} T(B) S = S^{-1} T(A) T(B) S = S^{-1} T(AB) S = T'(AB)$$

And the transformed matrices $T^{'}$ provide the new representation of the same group. The original representation and various representations obtained by similarity transformations choosing various matrices S differ only in that they are stated with respect to different frames and hence all such representation are said to be equivalent.

5.12 Reducible and irreducible representation:

A group of finite order may have two or more representations.From these two representations a single new representation may be formed ie., by combining the two matrices into one larger matrix.

From a representation $\{T^1(E), T^1(A), \cdots\}$ and a second representation $\{T^2(E), T^2(A), \cdots\}$ we can obtain a new representation consisting of larger matrices,

$$T(E) = \begin{pmatrix} \mathrm{T}^1(\mathrm{E}) & 0 \\ 0 & \mathrm{T}^2(\mathrm{E}) \end{pmatrix}, \quad T(A) = \begin{pmatrix} \mathrm{T}^1(\mathrm{A}) & 0 \\ 0 & \mathrm{T}^2(\mathrm{A}) \end{pmatrix}, \dots\dots\dots\dots$$

The matrix representation of the above form is said to be reducible ie., reducible representation can be expressed in terms of two or more representations.

The representation which cannot be expressed in terms other two or more representations is said to be irreducible. The irreducible representations of a group cannot be further reduced.

It is customary to indicate the structure of reducible representation by block form, the blocks representing the irreducible representations.

$T^1(\mathrm{A})$, $T^2(\mathrm{A})$ further reduced , this process can be carried on until we can find no unitary transformation which reduces all the matrices of a representation further. Thus the final form with all the matrices of T having the same reduced structure

$$T(\mathrm{A}) = \begin{bmatrix} T^1(\mathrm{A}) & & & & \\ & T^2(\mathrm{A}) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & T^s(\mathrm{A}) \end{bmatrix}$$

where $T^1(\mathrm{A})$, $T^2(\mathrm{A})$, $\cdots\cdots T^s(\mathrm{A})$ are called the irreducible representations. This cannot be further reduced.

5.13 Special Unitary Group:

A matrix A of order $m \times n$ is said to be unitary when it satisfies the relations $A A^{\dagger} = I_m$ and $A^{\dagger}A = I_n$ where I_m and I_n are unit matrices of order $m \times m$ and $n \times n$ respectively, $A^{\dagger}$ is transpose conjugate of A. A set of square unitary matrices of order $n \times n$ forms a group, denoted by $U(n)$, under matrix multiplication. It is known as a unitary group. A subgroup

$SU(n)$, of $U(n)$, is a set of special unitary matrices with determinant +1.

5.13.1 $SU(2)$ Group:

The $SU(2)$ Group is a group of 2×2 special unitary matrices under matrix multiplication with determinant +1.

Let u be a unitary matrix of order 2×2 , ie., $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then $u^{\dagger} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$

$\therefore \; uu^{\dagger} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} aa^* + bb^* & ac^* + bd^* \\ ca^* + db^* & cc^* + dd^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$

$aa^* + bb^* = 1$ (5.1)

$cc^* + dd^* = 1$ (5.2)

$ac^* + bd^* = 0$ (5.3)

$ca^* + db^* = 0$ (5.4)

As u belongs to $SU(2)$, the determinant of u must be equal to 5.1.

$|u| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1$ (5.5)

From equation 4, we get $d = -\frac{a^*}{b^*}\, c$ (5.6)

Substitute (6) in (5) we get $a(-\frac{a^*}{b^*}\, c) - bc = 1$

$- aa^* \dfrac{c}{b^*} - bc = 1$

$-(aa^* + bb^*)\frac{c}{b^*} = 1$ (5.7)

Put (5.1) in (5.7) we get $c = -\, b^*$ (5.8)

(5.8) in (5.6) we get $d = a^*$ (5.9)

Then the unitary matrix u with $|u| = 1$can be written as $u = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$

$|u| = aa^* + bb^* = 1$. $SU(2)$group may have the elements

$u_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $u_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

These elements form a group under matrix multiplication.

Problem:

Show that in general $SU(2)$ is not an abelian group.

Solution:

Let two elements of $SU(2)$ group are $u_1 = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ and $u_2 = \begin{bmatrix} c & d \\ -d^* & c^* \end{bmatrix}$

$$u_1 u_2 = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}\begin{bmatrix} c & d \\ -d^* & c^* \end{bmatrix} = \begin{bmatrix} ac - bd^* & ad + bc^* \\ -b^*c - a^*d^* & -b^*d + a^*c^* \end{bmatrix}$$

$$u_2 u_1 = \begin{bmatrix} c & d \\ -d^* & c^* \end{bmatrix}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} = \begin{bmatrix} ac - b^*d & bc + da^* \\ -ad^* - b^*c^* & -bd^* + a^*c^* \end{bmatrix}$$

$u_1 u_2 \neq u_2 u_1$ Hence in general $SU(2)$ is not an abelian group.

5.14 Special Orthogonal Group:

A matrix A of order $m \times n$ is said to be orthogonal when it satisfies the relations $A\,A^T = I_m$ and $A^T A = I_n$ where I_m and I_n are unit matrices of order $m \times m$ and $n \times n$ respectively, A^T is transpose of A. A set of square orthogonal matrices of order $n \times n$ forms a group, denoted by $O(n)$, under matrix multiplication. It is known as an orthogonal group. A subgroup (n) , of $O(n)$, is a set of special orthogonal matrices with determinant +1.

5.14.1 $SO(2)$ Group:

The $SO(2)$ Group is a group of 2×2 special orthogonal matrices under matrix multiplication with determinant +1.

Let u be a Orthogonal matrix of order 2×2 , ie., $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then $u^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\therefore \ uu^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} aa+bb & ac+bd \\ ca+db & cc+dd \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$

$aa + bb = 1$ (5.10)

$cc + dd = 1$ (5.11)

$ac + bd = 0$ (5.12)

$ca + db = 0$ (5.13)

As u belongs to $SO(2)$, the determinant of u must be equal to 1.

$|u| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1$ (5.14)

From equation 4, we get $d = -\frac{a}{b}\, c$ (5.15)

Substitute (6) in (5) we get $a(-\frac{a}{b}\, c) - bc = 1$

$$-\, aa^* \frac{c}{b^*} - bc = 1$$

$-(aa + bb)\frac{c}{b} = 1$ (5.16)

Put (5.10) in (5.16) we get $c = -\, b$ (5.17)

(5.17) in (5.15) we get $d = a$ (5.18)

Then the unitary matrix u with $|u| = 1$ can be written as

$u = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with $|u| = aa + bb = 1$.

When elements of $SU(2)$ matrices are real, $SO(2)$ and $SU(2)$ are the same.

Example:

Anticlockwise rotation about an axis (z-axis) is an example for $SO(2)$ group.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The angle θ is independent parameter and can assume various values and

$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$form a group under matrix multiplication. And $|R(\theta)| = 1$

The Identity element (unit Matrix) is obtained when $\theta = 0$.

$R(\theta_1)R(\theta_2)$ = $R(\theta_1 + \theta_2)$ it is the closure property of group

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

The inverse of $R(\theta)$ is $R(-\theta)$ = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$R(\theta_1)R(\theta_2)$ = $R(\theta_2)R(\theta_1)$ie., the group is abelian.

$[R(\theta_1)R(\theta_2)]R(\theta_3)$ = $R(\theta_1)[R(\theta_2)R(\theta_3)]$ie., associative law exists.

Problem:

Show that $SO(2)$ is always an abelian group.

Solution:

Let two elements of $SU(2)$ group are

u_1 = $\begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$ & u_2 = $\begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$$u_1u_2 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$u_2u_1 = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}\begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$u_1u_2 = u_2u_1$ Hence $SO(2)$ is always an abelian group.

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